이학박사 학위논문

장애물문제의 정칙성

(The Regularity of Obstacle Problems)

2019년 2월

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박진완의 이학박사 학위논문을 인준함

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The Regularity of Obstacle Problems

A dissertation
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Abstract

The Regularity of Obstacle Problems

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In this dissertation, we consider the regularity of solutions and the regularity of the free boundary of the obstacle problems. Specifically, we study the regularity of the free boundary of a non-convex fully nonlinear operator and the regularity of solutions and the free boundary of the double obstacle problem.

In order to prove the regularity of the free boundary of a non-convex fully nonlinear operator, we have the interior $C^{2,\alpha}$ regularity of the solution of the Dirichlet problem for the non-convex fully nonlinear operator. In the double obstacle problem for Laplacian, we use the ACF monotonicity formula and the Weiss' monotonicity formula. The monotonicity formulas are not applicable for the double obstacle problem for fully nonlinear operator. Hence, we exploit the fact that the term $\partial_e u/x_n$ is finite, where e is a direction orthogonal to e_n, for the global solution u with the half space function type upper obstacle $\psi = c(x_n^+)^2$.

Key words : Obstacle problem, Free boundary, Optimal($C^{1,1}$) regularity, non-convex fully nonlinear operator, double obstacle problem
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Contents

Chapter 1

Introduction

1.1 Introduction of Obstacle Problems

In this dissertation, we consider the regularity of solutions and the regularity of the free boundary of obstacle problems. Specifically, we study the regularity of the free boundary of a non-convex fully nonlinear operator and the regularity of solutions and free boundary of the double obstacle problem.

Free boundary problem is a world-widely active area of study and appear in various fields such as physics, biology and finance. Especially, obstacle problem which is a typical example of free boundary problem originates in fluid filtration in porous media, elsto-plasticity, optimal control and financial mathematics, see [Caf98, Fri].

In the classical obstacle problem, we consider the minimizer of the energy functional $J[u] = \int_D \frac{1}{2}|\nabla u|^2 + fudx$ over a function space $K = \{u \in W^{1,2}(D) \,|\, u = g \text{ on } \partial D, u \geq \phi \text{ in } D\}$, where $g \in W^{1,2}(D)$ and the *obstacle function* $\phi \in C^{2,\alpha}(D)$. Then the minimizer satisfies

$$\int_D \nabla u \cdot \nabla(v - u)dx \geq 0,$$

for any $v \in K$. Moreover, if $u \in H^2(D) \cap C(D)$, then u satisfies that

$$\begin{aligned} \Delta u &\leq 0 && \text{in } D, \\ u &\geq \phi && \text{in } D, \\ (\Delta u)(u - \phi) &= 0 && \text{in } D, \\ u &= g && \text{on } \partial D. \end{aligned} \tag{1.1}$$

The existence and uniqueness of $W^{2,p}(D)$ ($p > 2$) solution of (1.1) is obtained by the penalization method. The Laplacian Δu jumps along with the boundary of $\{u > \phi\}$ in D, $\partial\{u > \phi\} \cap D$ if $\Delta\phi < 0$ near the boundary. Hence the $C^{1,1}$ regularity is optimal for the solution u and the quadratic growth of u at a free boundary point x,

$$\sup_{B_r(x)} u - \phi \leq Cr^2,$$

implies the optimal regularity of the solution u.

Since we do not have the solution u in advance, so we also do not know where the boundary $\{u > \phi\} \cap D$ is. Hence the unknown boundary $\Gamma(u) := \partial\{u > \phi\} \cap D$ is called the *free boundary*.

Now, we are going to exlain the basic pattern to have the regularity of the free boundary. For simplicity, we exploit the reduced form of the classical obstacle problem (1.1):

$$\Delta v = f\chi_{\{v>0\}}, \quad v \geq 0 \quad \text{in } D, \tag{1.2}$$

where $v := u - \phi$ and $f := -\Delta\phi$. In order to have the regularity of the free boundary, we consider *rescaling function* of v at a free boundary point x_0, a point on the free boundary, $v_{r,x_0} := \frac{v(rx+x_0)}{r^2}$. Since v has the optimal ($C^{1,1}$) regularity, $C^{1,1}$ norm of v_{r,x_0} are bounded. Hence v_{r,x_0} converges in $C^{1,\alpha}$ to a limit function v_0 as $r \to 0$. The limit function is called *blowup*. With thickness assumption of zero set $\{v = 0\}$ the blowup function should be a half space solution, $v_0 = c(x_n^+)^2$, for an appropriate coordinate system. Then by using the C^1 closeness between u_r to u_0 for sufficiently small r, we have the Lipschitz and C^1 regularity of the free boundary.

The regularity of the free boundary of the classical obstacle problem is first proved by [Caf77]. Thereafter, the existence and uniqueness of the solution, $C^{1,1}$ regularity of the solution and C^1 (and higher) regularity of the free boundary of various obstacle problems for linear operator have been studied by numerous researcher.

1.2 A Preview of Dissertation

The obstacle problem for the fully nonlinear operator is first researched by [Lee98] and a general class of obstacle problem for elliptic and parabolic fully nonlinear operator is considered by [FS14]. In those papers, it is assumed that the fully nonlinear operator $F(\cdot)$ is convex with respect to the matrix variable. The first reason why the convexity is needed is that the convexity implies interior $C^{2,\alpha}$ regularity of the viscosity solution of $F(D^2u) = 0$, which is known by the Evans-Krylov theorem. The second reason is the second directional derivative of the solution, u_{ee} with respect to any direction e is a solution of a linear differential equation. Hence, the lack of convexity can be the main difficulties when one attempts to study the regularity of the free boundary for non-convex fully nonlinear operators.

In Chapter 3, we suggest a specific non-convex assumption on $F(\cdot)$ and show that the difficulties can be resolved for the non-convex operator. Finally, we obtain the regularity of the free boundary of the obstacle problem for fully nonlinear operator with the non-convex assumption. As far as we know, this paper is the first result on the obstacle problem for a non-convex fully nonlinear operator.

In the double obstacle problem, one consider a energy minimizer of the energy functional $J[u] = \int_D \frac{1}{2}|\nabla u|^2 dx$ on a function space $K = \{u \in W^{1,2}(D) \,|\, u = g \text{ on } \partial D, \phi_2 \le u \le \phi_2 \text{ in } D\}$. Then the solution satisfies

$$\begin{cases} \Delta u \ge 0, & \text{in } \{u > \phi_1\} \cap D, \\ \Delta u \le 0, & \text{in } \{u < \phi_2\} \cap D, \\ \phi_1(x) \le u(x) \le \phi_2(x) & \text{in } D, \\ u(x) = g(x) & \text{on } \partial D, \end{cases} \tag{1.3}$$

with $\phi_1, \phi_2 \in C^{1,1}(\overline{D})$, $\partial D \in C^{2,\alpha}$, $g \in C^{2,\alpha}(\overline{D})$ and $\phi_1 \le g \le \phi_2$ in ∂D. The optimal regularity for the solution of the double obstacle problem is $C^{1,1}$, like the classical obstacle problem. On the other hand, the double obstacle problem has two free boundaries, $\partial\{u > \phi_1\} \cap D$ and $\partial\{u < \phi_2\} \cap D$. For simplicity, we introduce the reduced problem for the double obstacle problem (1.3) is as follows :

$$F(D^2u, x) = f\chi_{\{0<u<\psi\}} + F(D^2\psi, x)\chi_{\{0<u=\psi\}} \quad \text{a.e. in } D \tag{1.4}$$

with $0 \le u \le \psi$ in D, $f \in L^\infty(D)$ and $\psi \in C^{1,1}(\overline{D})$, see Subsection 5.1.1.

In [Ale], the author presented global solutions of the double obstacle problem in 2 dimension with C^2-obstacles whose free boundaries have a corner. This fact implies that the free boundaries of the blowup function of a solution u could have a corner, where u is a solution of the double obstacle problem (1.3) with C^2-obstacles. Then the regularity of the free boundaries could not be better than the Lipschitz regularity. Thus in Chapter 4 and 5, we study the regularity of the free boundaries of the problem with $C^{1,1}$-obstacles.

In Chapter 5, we prove the existence and uniqueness of $W^{2,p}$ $(n < p < \infty)$ solution of the *double obstacle problem* for fully nonlinear operator with $C^{1,1}$ obstacles, ϕ_1 and ϕ_2. Furthermore, we prove the optimal $(C^{1,1})$ regularity of solutions the double obstacle problems.

In Chapter 4 and 5, we prove the regularity of the free boundaries for double obstacle problem for Laplacian and fully nonlinear operator. The main idea for the study of the regularity of the free boundaries is that considering the $C^{1,1}$ upper obstacle ψ in (1.4) as a solution of the reduced classical obstacle problem, (1.2). The essential approaches to prove the regularity of the free boundaries for Laplacian and fully nonlinear operator are different, so we introduce the two distinct methods in Chapter 4 and 5.

Indeed, For the Laplace case, we use the ACF monotonicity formula and Weiss' monotonicity formula which is not applicable for the fully nonlinear operator. Hence, we focus on the fact that with the thickness assumption for the upper obstacle ψ, the blowup of ψ is a half space type function, $c(x_n^+)^2$. Then, the term $\partial_e u_0/x_n$ is finite, where e is a direction orthogonal to e_n, u_0 is a blouup of u the upper obstacle ψ with then thickness assumption for ψ. That implies that u_0 is also of the form of a half space type function and the regularity of the free boundaries.

Chapter 2

Preliminaries

In this chapter, we introduce the concept of viscosity solution and theorems for regularity of solutions of PDEs of fully nonlinear operator which are used in the following chapters. We also explain the definitions of useful concept to study regularity of the free boundary of obstacle problem such as rescaling, blowup and thickness assumption.

2.1 Fully Nonlinear Operator

The concept of viscosity solution for the fully nonlinear operator is developed by [CL] in the research on Hamilton-Jacobi equations. There are important fully nonlinear second order operators such as Hamilton-Jacobi-Bellman equation, Isaacs equation, the Monge-Ampère equation and so on. Hamilton-Jacobi-Bellman equation arises in the optimal cost in a stochastic control problem. Isaacs equations is a non-convex operator and originates in differential games.

2.1.1 Viscosity Solutions

Definition 2.1.1. We say F is uniformly elliptic if there are *elliptic constants* $0 < \lambda \leq \Lambda < \infty$ such that for any real $n \times n$ symetric matrices and $x \in D$

$$\lambda \|N\| \leq F(M+N, x) - F(M, x) \leq \Lambda \|N\|,$$

for any N, where N is a non-negative definite symmetric matrix.

Definition 2.1.2. A continuous function u in D is said to be a viscosity subsolution (supersolution) to $F(D^2u(x), x) = f(x)$ in D, when the following condition holds: if $x_0 \in D, \phi \in C^2(D)$ and $u - \phi$ has local maximum at x_0 then

$$F(D^2\phi(x_0), x_0) \geq (\leq) f(x_0).$$

Furthermore, A continuous function u in D is said to be a viscosity solution of $F(D^2u, x) = f(x)$ in D, if it is a viscosity subsolution and supersolution.

Let $0 \leq \lambda \leq \Lambda$. For $M \in \mathcal{S}$, where $\mathcal{S}$ is the space of real $n \times n$ symmetric matrices, we define

$$\mathcal{M}^-(M, \lambda, \Lambda) = \mathcal{M}^-(M) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

$$\mathcal{M}^+(M, \lambda, \Lambda) = \mathcal{M}^+(M) = \lambda \sum_{e_i<0} e_i + \Lambda \sum_{e_i>0} e_i,$$

where $e_i = e_i(M)$ are the eigenvalues of M.

Definition 2.1.3. Let f be a continuous function in D and $0 < \lambda \leq \Lambda < \infty$. The space of continuous function u in D, $\underline{S}(\lambda, \Lambda, f)$ consist of all functions such that $\mathcal{M}^+(D^2u, \lambda, \Lambda) \geq f(x)$ in the viscosity sense in D. In the same way, $\overline{S}(\lambda, \Lambda, f)$ is defined by the space of continuous functions u in D such that $\mathcal{M}^-(D^2u, \lambda, \Lambda) \leq f(x)$ in the viscosity sense in D. we also define

$$S(\lambda, \Lambda, f) = \underline{S}(\lambda, \Lambda, f) \cap \overline{S}(\lambda, \Lambda, f),$$

$$S^*(\lambda, \Lambda, f) = \underline{S}(\lambda, \Lambda, -|f|) \cap \overline{S}(\lambda, \Lambda, |f|),$$

2.1.2 Regularity of the Solution of the Fully Nonlinear Operator

In this subsection, we introduce some theorem for the regularity of solutions of $F(D^2u) = 0$. Theorem (2.1.1) is about the $C^{2,\alpha}$ regularity of the solution of $F(D^2u) = 0$, where F is convex and this regularity result is used in many steps in the proof of the regularity of the free boundary. For example, it plays an essential role in the proof of the fact that blowup functions of solutions of the obstacle problems still are solutions of obstacle problems. In Chapter 3, we are going to deal with the obstacle problem for a non-convex fully nonlinear operator. Hence we require to have any substitute of Theorem (2.1.1) for the non-convex fully nonlinear operator.

The substitute is the non-convex operator version of Theorem (2.1.5). In [CY], the authors have the a priori interior $C^{2,\alpha}$ regularity of the solution of $F(D^2u) = 0$ for the smooth non-convex fully nonlinear operator F. Then, we could have Theorem (2.1.5) for the non-convex operator, by using the method of continuity.

The theorem implies that the solution u of the obstacle problem has $C^{2,\alpha}$ regularity in the set, $\{u > \phi\}$. Specifically, we know that $F(D^2u) = 0$ on $\{u > \phi\}$ and the solutions of the obstacle problem has $W^{2,p}$ regularity. Thus, for the Dirichlet problem for the small ball in $\{u > \phi\}$ the boundary condition is always satisfied and the interior $C^{2,\alpha}$ regularity of the solution of the obstacle problem in $\{u > 0\}$ is obtained by the theorem. This implies the fact that blowup functions of solutions of the obstacle problems still are solutions of obstacle problems (see Lemma 3.3.3) and it plays important role in the proof of the regularity of the free boundary, see Proposition 3.3.5.

Theorem 2.1.1. *([CC]) Let F be convex and u be a viscosity solution of $F(D^2u) = 0$ in B_1. Then $u \in C^{2,\alpha}(\overline{B}_{1/2})$ and*

$$\|u\|_{C^{2,\alpha}(\overline{B}_{1/2})} \leq C(\|u\|_{L^\infty(B_1)} + |F(0)|),$$

where $0 < \alpha < 1$ and C are universal constants.

Proposition 2.1.2. *([CC]) Let $0 < \alpha < 1$ and $u \in C^{2,\alpha}(D)$ be a solution of*

$$F(D^2u, x) = f(x) \quad \text{in } D.$$

Assume that $F \in C^\infty(\mathcal{S} \times D)$ and $f \in C^\infty(D)$. Then $u \in C^\infty(D)$.

Theorem 2.1.3. *([CC]) Let F be convex and smooth and g be a smooth function in $\overline{B}_1$. Then there exists a universal $\alpha \in (0,1)$ such that if $u \in C^{2,\alpha}(\overline{B}_1)$ is a solution of*

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1, \\ u = g & \text{on } \partial B_1 \end{cases}$$

then

$$\|u\|_{C^{2,\alpha}(\overline{B}_1)} \leq C\left(\|g\|_{C^3(\overline{B}_1)} + |F(0)|\right),$$

where C is a universal constant.

Theorem 2.1.4. *([CC]) Let F be convex and smooth, and $g \in C^\infty(\overline{B}_1)$. Then there exists a unique viscosity solution $u \in C(\overline{B}_1)$ th the Dirichlet problem*

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1, \\ u = g & \text{on } \partial B_1. \end{cases} \tag{2.1}$$

Futhermore, $u \in C^\infty(\overline{B}_1)$ and

$$\begin{aligned} \|u\|_{C^{2,\alpha}(\overline{B}_1)} &\le C_1 \left(\|g\|_{C^3(\overline{B}_1)} + |F(0)|\right), \\ \|u\|^*_{C^{2,\alpha}(\overline{B}_{r/2})} &\le C_2 \left(\|u\|_{L^\infty(B_r)} + r^2|F(0)|\right), \end{aligned} \tag{2.2}$$

for any ball $B_r = B_r(x_0) \subset B_1$, where $0 < \alpha < 1$, C_1 and C_2 are universal constants. Here $\|\cdot\|^*_{C^{2,\alpha}(\overline{B}_d)}$ denotes the adimensional $C^{2,\alpha}(\overline{B}_d)$-norm.

Proposition 2.1.5. *([CC]) Let F be a convex and $g \in C(\partial B_1)$. Then there exists a unique viscosity solution $u \in C(\overline{B}_1)$ to the Dirichlet problem* (2.1). *Furthermore, $u \in C^{2,\alpha}(B_1)$ and satisfies* (2.2) *for any $B_r = B_r(x_0) \subset B_1$.*

2.2 Rescaling, Blowup and Thickness assumption.

In this section, we introduce the rescaling, blowup and thickness assumption. The concepts are essential to study the regularity of the obstacle problems and they are applied in the following chapters. In order to give a general explanation, we will use u to denote a solution of obstacle problems containing classical obstacle problem, obstacle problem for fully nonlinear operator, double obstacle problem and so on.

Let us start with an introduction of a basic process to prove the regularity of the free boundary of obstacle problems using the concepts. The rescaling function of the solution u at a free boundary point x_0 is defined by $u_r = u(rx+x_0)/r^2$. The optimal regularity of the solution implies that there exists a limit function u_0 of rescaling function u_r, where $r \to 0$. The limit function is called the blowup. The thickness assumption of the zero set of the solution u implies that the blowup function u_0 of the solution u is the half space solution, $c(x_n^+)^2$. Then for a cone C with axis parallel to e_n, if a direction $e \in C$, then the directional derivative with respect to e of the blowup u_0 is positive in $\mathbb{R}^n$.By using the fact that the rescaling function u_r converges to u_0 as $r \to 0$ in $C^{1,\alpha}$, it is obtained that $\partial_e u$ is also positive, near the free boundary point. It is called the directional monotonicity and then we have Lipschitz and C^1 regularity of the free boundary.

Definition 2.2.1. Let u be a solution of the obstacle problems in B_r. Then a *rescaling function* of u at x_0 with $\lambda > 0$ is

$$u_\lambda(x) = u_{\lambda,x_0}(x) := \frac{u(x_0 + \lambda x) - u(x_0)}{\lambda^2}, \quad x \in B_{r/\lambda}.$$

The $C^{1,1}$-regularity of solution u implies the uniform boundedness of $C^{1,1}$-norm of the rescaling functions and the uniform boundedness gives limit functions which are called a blowup and a shrink-down. More precisely, if u is a solution of the obstacle problems in B_r, then for a sequence $\lambda_i \to 0$, there exists a subsequence λ_{i_j} of λ_i and $u_0 \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that

$$u_{\lambda_{i_j}} \to u_0 \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^n) \quad \text{for any } 0 < \alpha < 1.$$

Such u_0 is called a *blowup of u at x_0*.

Definition 2.2.2. We denote by $\delta_r(u, x)$ the thickness of $\Lambda(u)$ on $B_r(x)$, i.e.,

$$\delta_r(u, x) := \frac{\mathrm{MD}(\Lambda(u) \cap B_r(x))}{r},$$

where $\mathrm{MD}(A)$ is the least distance between two parallel hyperplanes containing A. We will use the abbreviated notation $\delta_r(u)$ for $\delta_r(u, 0)$.

Remark. The thickness δ_r satisfies $\delta_1(u_r) = \delta_r(u)$, where $u_r = u_{r,0}$. Thus, by the fact that $\limsup_{r\to 0} \Lambda(u_r) \subset \Lambda(u_0)$, we have

$$\limsup_{r\to 0} \delta_r(u) \le \delta_1(u_0).$$

Hence the thickness assumption $\delta_r(u) \ge 0$ for $r < r_0$ implies

$$\delta_r(u_0) \ge \epsilon_0 \quad \forall r > 0,$$

and this implies that u_0 is a half space solution $u_0 = c(x_n^+)^2$.

In the following sections, we define solution spaces, $P_r(M)$ and $P_\infty(M)$. Basically, $u \in P_r(M)$ means that u is a local solution, a solution in B_r and $u \in P_\infty(M)$ represents that u is a global solution, a solution in $\mathbb{R}^n$. The solution spaces will be defined differently, in every obstacle problem, studied in following sections.

Chapter 3

Obstacle Problem for a Non-convex Fully Nonlinear Operator

3.1 Introduction

In [Lee98], the existence and uniqueness of the solution and $C^{1,1}$ regularity of the solution for the obstacle problem for fully nonlinear operator in a domain $\Omega \subset \mathbb{R}^n$,

$$\begin{aligned} F(D^2u) &\leq 0 \quad \text{in } \Omega, \\ u &\geq \phi \quad \text{in } \Omega, \\ (F(D^2u))(u-\phi) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{3.1}$$

with $\phi \in C^{2,\alpha}(\overline{\Omega})$ and $0 \geq \phi$ on $\partial\Omega$ is proved, for convex operator $F(\mathcal{M})$ (see Theorem 1.4, Theorem 1.5 and Theorem 2.2 in [Lee98]). In addition, for convex operator $F(\mathcal{M})$, it is proved that the free boundary of the problem (3.1) has C^1 regularity near a free boundary point under a thickness assumption by Lee, see Theorem 3.2 in [Lee98]. Moreover, the obstacle problem for the Monge-Ampère equation is studied by [Lee01, Sav].

Hence, the open question is the obstacle problem for fully nonlinear operator with assumptions for F, which includes a non-convex case. Thus, in this section, we are going to prove the regularity of the free boundary of the obstacle problem for fully nonlinear operator, (3.1) under specific conditions for the operator and level sets of the operator.

There are two main difficulties in our work. The first is the fact that we need to know Evans-Krylov type estimate for the general fully nonlinear operator, since it is necessary in the proof of existence of the solution of (3.1) and the method of blowup. The second difficulty is that it is not clear that the second derivative with respect to any direction e, u_{ee}, is a supersolution of a linear operator. For convex operator $F(\mathcal{M})$, the second derivative, u_{ee}, is a supersolution of a linear operator and it plays a key role in the proof of the regularity of the free boundary, see Lemma 3.7 of [Lee98] and Proposition 3.2 of [FS14]. However, without the convexity assumption for fully nonlinear operator, $F(\mathcal{M})$, we do not know that u_{ee} is a supersolution of a linear operator. Thus we need to consider a general condition for fully nonlinear operator which has a substitute for $Lu_{ee} \leq 0$, where L is a linear operator.

In the rest of this introduction, we discuss the difficulties in detail and summarize the contents in this section. As described in the paragraph above, Evans-Krylov type theorem for fully nonlinear operator F is essential to discuss the regularity of the free boundary of the obstacle problem for the operator. Thus, we have looked for a non-convex fully nonlinear operator which has Evans-Krylov theorem, $C^{2,\alpha}$ regularity for solutions of $F(D^2u) = 0$ and an interior $C^{2,\alpha}$ estimate of solutions. Finally, we focus on the conditions (1) and (2) in Theorem 1 of [CY], which are the same as (E4') and (E5') in our paper, see Subsection 3.1.2. The conditions above allow us to consider the fully nonlinear operators including non-convex case.

In Section 3.2, by using techniques in [CC] and theorems in [CY], we discuss an Evans-Krylov type theorem for fully nonlinear operator F under the condition (E4), (E5) and other specific conditions. More specifically, we prove the existence and uniqueness of $C^{2,\alpha}$ (even C^∞) solution of the Dirichlet problem,

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1, \\ u = g & \text{on } \partial B_1, \end{cases}$$

and have an interior $C^{2,\alpha}$ regularity for the solution of the Dirichlet problem under the conditions, see Theorem 3.2.2 and Remark 3.1.4.

Thus, the existence and uniqueness of the solution and $C^{1,1}$ regularity of solution of problem (3.1) with the non-convex conditions are proved by the same argument in [Lee98]. Therefore, in the rest of this section, we study the regularity of the free boundary of (3.1) with the non-convex conditions.

In Subsection 3.3.2, we prove that global solutions are convex and global solutions with a thickness assumption are half-space solutions, see Subsection 3.1.3, Proposition 3.3.5 and Proposition 3.3.7. Once we prove Proposition 3.3.7, then the proof of Theorem 3.1.1, which is the main theorem of this section, follows a typical process, i.e., it is almost the same as that for Laplace operator and convex fully nonlinear opeartor in [Fri], [Lee98], [PSU], [FS14], [IM16a] and [LPS]. Thus, in Subsection 3.3.3 and 3.3.4, we follow the similar methods to prove Theorem 3.1.1.

We note that in theorems in this section, we do not assume the conditions (E4') and (E5'), which are the same as the conditions in [CY]. Instead, we assume the conditions (E4) and (E5) in Theorems of this section. That is because, under the conditions (E4) and (E5), $-e^{-Ka_{ij}D_{ij}u}$ is a supersolution of a linear operator, where u is a solution of $F(D^2u) = 0$, see Lemma 3.3.4. It is an alternative to the fact that u_{ee} is a supersolution of a linear operator, where u is a solution of $F(D^2u) = 0$, for the convex fully nonlinear operator, see Proposition 3.3.5 of our paper, Theorem 1.4 of [LS01] and Proposition 3.2 of [FS14].

3.1.1 Notation

We will use the following notations throughout this chapter.

C, C_0, C_1	generic constants
χ_S	the characteristic function of a set S, $(S \subset \mathbb{R}^n)$
$\overline{S}$	the closure of a set S
∂S	the boundary of a set S
$\lvert S \rvert$	$n-$ dimensional Lebesgue measure of a set S
$B_r(x), B_r$	$\{y \in \mathbb{R}^n : \lvert y - x \rvert < r\}, \quad B_r(0)$
$\Omega(u)$	$\{u > \phi\}$
$\Lambda(u)$	$B_1 \setminus \Omega(u)$, the coincident set
$\Gamma(u)$	$\partial\Lambda(u) \cap B_1 = \partial\Omega(u) \cap B_1$
u^+, u^-	$\max(u, 0), \max(-u, 0)$
$\lVert\mathcal{M}\rVert$	$\sup_{x \in \partial B_1} \lvert\mathcal{M}x\rvert$ where $\mathcal{M}$ is a symmetric $n \times n$ matrix
$P_r(M), P_\infty(M)$	See Definition 3.1.1, 3.1.2
$\delta_r(u, x), \delta_r(u)$	See Definition 3.1.3

3.1.2 The Conditions on Fully Nonlinear Operator and Level Sets

We introduce the following conditions for F which include a non-convex case. This type of conditions first appeared in [CY], which is the paper about a priori $C^{2,\alpha}$ interior estimate of the solution of $F(D^2u) = 0$ with convex level set.

We note that there is an example of the operator which satisfies the conditions (E1)-(E3), (E4') and (E5') in [CY]. Furthermore, for dimension 3, the fully nonlinear operator,

$$F(D^2u) = \arctan\lambda_1 + \arctan\lambda_2 + \arctan\lambda_3 = c$$

with $|c| \geq \pi/2$ satisfies the condition (E4'), where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of D^2u, see [CNS] and [Yua].

Conditions on $F = F(\mathcal{M})$:

(E1) $F(0) = 0$ and $F \in C^\infty(\mathbb{R}^{n\times n})$.

(E2) F is uniformly elliptic: there are two positive constants $0 < \lambda \leq \Lambda < +\infty$ such that

$$\lambda\,\|\mathcal{N}\| \leq F(\mathcal{M}+\mathcal{N}) - F(\mathcal{M}) \leq \Lambda\,\|\mathcal{N}\|$$

holds for any symmetric $n\times n$ matrix $\mathcal{M}$ and a positive definite symmetric $n\times n$ matrix $\mathcal{N}$.

(E3) F is convex, concave or close to linear function near infinity: there exists $\delta = \delta(n,\lambda,\Lambda)$ and $\chi > 0$ such that $F(\mathcal{M})$ is convex or concave in $\{\mathcal{M} \in \mathbb{R}^{n\times n} \mid \|\mathcal{M}\| \geq \chi\}$ or

$$\left\|D^2F(\mathcal{M})\right\| \leq \frac{\delta}{\|\mathcal{M}\|} \quad \text{for } \|\mathcal{M}\| \geq \chi.$$

Conditions on the Level Sets: For any positive constant $t > 0$, there exist $\omega(t)$ and $\Theta(t)$ such that for $\mathcal{M}_0 \in \Sigma = \{\mathcal{M} \in \mathbb{R}^{n\times n} \mid F(\mathcal{M}) = 0\}$ with $F_{ij,kl}(\mathcal{M}_0) \not\geq 0$ and $\|\mathcal{M}_0\| \leq t$, the following relationships hold:

(E4)

$$\mathrm{II}(\mathcal{M}_0) \leq -\omega(t) < 0,$$

where $\mathrm{II}(\mathcal{M}_0)$ is the second fundamental form of $\Sigma \cap \{\mathcal{M} \in \mathbb{R}^{n\times n} \mid tr\mathcal{M} = tr\mathcal{M}_0\}$ at $\mathcal{M}_0$,

(E5)

$$\measuredangle(I, F_{ij}(\mathcal{M}_0)) \geq \Theta(t) > 0 \quad \text{and} \quad \measuredangle(-I, F_{ij}(\mathcal{M}_0)) \geq \Theta(t) > 0$$

where $\measuredangle(I, F_{ij}(\mathcal{M}_0))$, $\measuredangle(-I, F_{ij}(\mathcal{M}_0))$ are the angles between the identity matrix I and $(F_{ij}(\mathcal{M}_0))$, $-I$ and $(F_{ij}(\mathcal{M}_0))$, respectively, where $(F_{ij}(\mathcal{M}_0))$ is the normal to Σ at $\mathcal{M}_0$.

For any $t > 0$, there exist $\omega(t)$ and $\Theta(t)$ such that for $\mathcal{M}_0 \in \Sigma = \{\mathcal{M} \in \mathbb{R}^{n\times n} \,|\, F(\mathcal{M}) = 0\}$ with $F_{ij,kl}(\mathcal{M}_0) \nleq 0$ and $\|\mathcal{M}_0\| \leq t$, the following relationships hold:

(E4')

$$\mathrm{II}(\mathcal{M}_0) \geq \omega(t) > 0,$$

(E5')

$$\measuredangle(I, F_{ij}(\mathcal{M}_0)) \geq \Theta(t) > 0 \quad \text{and} \quad \measuredangle(-I, F_{ij}(\mathcal{M}_0)) \geq \Theta(t) > 0.$$

3.1.3 Definitions

Since we focus on the regularity of the free boundary near a free boundary point and we are going to use method in [LS01, FS14, IM16a], we deal with a localized version of (3.1),

$$\begin{aligned} F(D^2u) &\leq 0 \quad \text{in } B_r,\\ u &\geq \phi \quad \text{in } B_r,\\ (F(D^2u))(u-\phi) &= 0 \quad \text{in } B_r, \end{aligned} \tag{3.2}$$

with $\phi \in C^{2,1}(\overline{B_r})$ and $0 \in \Gamma(u)$. In addition, the optimal regularity ($C^{1,1}$ regularity) of the solution u for the obstacle problem (3.1) for fully nonlinear operator is already obtained in [Lee98], see Theorem 2.2 of [Lee98]. Thus, we define the space of local solutions as follows:

Definition 3.1.1. (Local solutions) For a given positive constant M, let $P_r(M)$ be a class of solutions $u \in C^{1,1}(B_r)$ of (3.2) with $\phi \in C^{2,1}(\overline{B_r})$, $0 \in \Gamma(u)$ and $\left\|D^2u\right\|_{L^\infty(B_r)} \leq M$.

For simplicity we consider $v := u - \phi$, where $u \in P_r(M)$. Then v is a solution of the obstacle problem with zero obstacle,

$$\tilde{F}(D^2v, x) = f(x)\chi_{\{v>0\}}, \quad v \geq 0 \quad \text{in } B_r,$$

where $\tilde{F}(\mathcal{M}, x) := F(\mathcal{M}+D^2\phi)-F(D^2\phi)$, $f(x) := -F(D^2\phi)$ and $\tilde{F}(\mathcal{M}, x), f(x) \in C^{0,1}(B_r)$ with respect to x.

In order to utilize the method of blowup, we consider the rescaling function and the blowup. The *rescaling function* of $v := u - \phi$ at $0 \in \partial\{v > 0\} \cap B_r$ for $\rho > 0$ is

$$v_\rho(x) := \frac{v(\rho x)}{\rho^2}, \quad x \in B_{r/\rho}.$$

$C^{1,1}$ regularity of the solution v implies the uniform boundedness of $C^{1,1}$ norm of the rescaling functions v_ρ and the uniform boundedness gives existence of limit function called *blowup*. More specifically, for a sequence $r_i \to 0$, there exists a subsequence r_{i_j} of r_i such that

$$v_{r_{i_j}} \to v_0 \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^n) \quad \text{for any } 0 < \alpha < 1.$$

Such v_0 is called a blowup of v at 0.

In Section 3.3, we are going to prove that the blowup v_0 of v is a solution of the obstacle problem with zero obstacle,

$$\tilde{F}(D^2 v_0, 0) = f(0)\chi_{\{v_0>0\}}, \quad v_0 \geq 0 \quad \text{a.e. in } \mathbb{R}^n,$$

under some conditions for the fully nonlinear operator, see Lemma 3.3.3. Therefore, we define the space of global solutions as follows:

Definition 3.1.2. (Global solutions) For a given positive constant M, let $P_\infty(M)$ be a class of solutions $u \in C^{1,1}_{loc}(\mathbb{R}^n)$ of

$$F(D^2u) = \chi_{\{u>0\}}, \quad u \geq 0 \quad \text{a.e. in } \mathbb{R}^n.$$

with $0 \in \Gamma(u)$ and $\left\|D^2u\right\|_{L^\infty(\mathbb{R}^n)} \leq M$.

For $u \in P_\infty(M)$, we are going to consider another limit function u_∞ of rescaling function u_r, when r goes to ∞, i.e., for a sequence $r_i \to \infty$, there exists a subsequence r_{i_j} of r_i such that

$$u_{r_{i_j}} \to u_\infty \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^n) \quad \text{for any } 0 < \alpha < 1.$$

Such u_∞ is called a *shrink-down* of u at 0.

Definition 3.1.3. The *thickness* of $\Lambda(u)$ on B_r is defined by

$$\delta_r(u, x^0) = \frac{\mathrm{MD}(\Lambda(u) \cap B_r(x^0))}{r},$$

where $\mathrm{MD}(A)$ is the least distance between two parallel hyperplanes enclosing $A \subset \mathbb{R}^n$.

In Theorem 3.1.1, to obtain the regularity of free boundary near 0, we assume a thickness assumption for the coincident set of solution $u \in P_r(M)$,

$$\delta_r(u) = \delta_r(u, 0) \geq \epsilon_0 > 0 \quad \forall r < 1/4.$$

For $v = u - \phi$, we will use the notations $\Omega(v)$, $\Lambda(v)$ and $\Gamma(v)$ for $\{v > 0\} \cap B_r = \{u > \phi\} \cap B_r$, $\{v = 0\} \cap B_r$ and $\partial\Omega(v) \cap B_r$, respectively. Moreover, we define

$$\delta_r(v) = \frac{\mathrm{MD}(\Lambda(u) \cap B_r)}{r}.$$

Then the above thickness assumption implies

$$\delta_r(v) \geq \epsilon_0 > 0 \quad \forall r < 1/4$$

and

$$\delta_r(v_0) = \frac{\mathrm{MD}(\Lambda(v_0) \cap B_r)}{r} \geq \epsilon_0 > 0 \quad \forall r > 0,$$

where v_0 is a blowup of $v = u - \phi$ at 0.

3.1.4 Main Theorems

The goal of this section is obtaining the regularity of the free boundary of the obstacle problem for fully nonlinear operator under the conditions which allow of a non-convex case. The main theorem of the section is as follows:

Theorem 3.1.1. *Let $u \in P_1(M)$ with the obstacle function ϕ such that $-F(D^2\phi) \geq c > 0$ in B_1 and assume that $F(\mathcal{M})$ satisfies $(E1)$-$(E3)$ and*

$$\Sigma_t := \{\mathcal{M} \,|\, tF(\mathcal{M}) + (1-t)tr(\mathcal{M}) = 0\}$$

satisfies $(E4)$ and $(E5)$ for all $t \in [0, 1]$ with $\omega_t(C_1), \Theta_t(C_1)$, where C_1 is in the universal constant in Theorem 3.2.1. We further assume that there are functions $h, k : \mathbb{R} \to \mathbb{R}$ such that

$$\omega_t(C_1) \leq h(t) \in o(t^{1/2}), \quad \Theta_t(C_1) \leq k(t) \in o(t)$$

and

$$\delta_r(u) \geq \epsilon_0 > 0, \quad \forall r < 1/4.$$

Then for some

$$\tilde{r} = \tilde{r}\left(u, c, \|DF\|_{L^\infty(B_{E+\|F(D^2\phi)\|_{L^\infty(B_1)}})}, \left\|F(D^2\phi)\right\|_{L^\infty(B_1)}, M, \lambda, \Lambda\right) > 0,$$

$\Gamma(u) \cap B_{\tilde{r}}(0)$ *is a* C^1 *surface.*

Remark. We note that the conditions for Σ_t, $t \in [0,1]$, $\omega_t(C)$ and $\Theta_t(C)$ are only for the Evans-Krylov type theorem, see Remark 3.2 and Theorem 3.2.2. Also note that $\omega_0(C)$, $\Theta_0(C)$ are zero, since $\Sigma_0 = \{\mathcal{M} \,|\, tr(\mathcal{M}) = 0\}$ is a hyperplane in $\mathbb{R}^{n\times n}$ which is perpendicular to the identity matrix I.

Corollary 3.1.2. *Let u, ϕ, F and Σ_t be as in Theorem 3.1.1. Then the following hold:*

(i) $\Gamma(u)$ *is* $C^{1,\alpha}$ *surface for any* $0 < \alpha < 1$ *in a neighborhood of* 0.

(ii) If further $D^2\phi \in C^{m,\beta}$ $(m \geq 1, 0 < \beta < 1)$ *in a neighborhood of* 0*, then* $\Gamma(u)$ *is* $C^{m+1,\beta}$ *surface in some neighborhood of* 0.

(iii) If F and ϕ is analytic in a neighborhood of 0*, then* $\Gamma(u)$ *is analytic in some neighborhood of* 0.

Remark. By using Theorem 3.2.2 in our paper and Theorem 8.1 of [CC], one can have $C^{2,\alpha}$ regularity of solutions for $F(D^2u, x) = f(x)$, under specific conditions for the operator, zero sets of the operator and the forcing term. Furthermore, the optimal regularity of solutions of the obstacle problem

$$F(D^2u, x) = f(x)\chi_{\{u>0\}}, \quad u \geq 0 \quad \text{in } B_r, \tag{3.3}$$

can be obtained by the method in [FS14, IM16a]. Then, one can also discuss the regularity of the free boundary for (3.3) under the conditions. In other words, the proof for the regularity of the free boundary for (3.3) naturally follows the method for Theorem 3.1.1, which is the theorem for the regularity of the free boundary for (3.2).

3.2 $C^{2,\alpha}$ Regularity of Solutions for $F(D^2u) = 0$

Theorem 3.2.1. *Suppose* $F \in C^\infty(\mathbb{R}^{n\times n})$ *satisfies* $(E2)$ *and* $\Sigma = \{\mathcal{M} \mid F(\mathcal{M}) = 0\}$ *satisfies* $(E4)$ *and* $(E5)$ *and let* $g \in C^3(\overline{B_1})$. *Then if* $u \in C^{2,\alpha}(\overline{B_1})$ *is a solution of*

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1, \\ u = g & \text{on } \partial B_1, \end{cases}$$

then

$$\left\|D^2u\right\|_{L^\infty(B_1)} \leq C_1,$$

for a universal constant C_1 *and*

$$\|u\|_{C^{2,\alpha}(\overline{B_1})} \leq C(K)\left(\|g\|_{C^3(\overline{B_1})} + |F(0)|\right),$$

for a universal constant $\alpha \in (0,1)$, *where*

$$\begin{aligned} K &= K\left(n, \lambda, \Lambda, \omega(C_1), \Theta(C_1), \left\|D^2F\right\|_{L^\infty(B_{C_1})}, \|g\|_{C^3(\overline{B_1})}\right) \\ &= C\left(n, \lambda, \Lambda, \|g\|_{C^3(\overline{B_1})}\right)\left(\frac{\left\|D^2F\right\|^2_{L^\infty(B_{C_1})}}{\omega(C_1)\left|\tan\left(\|g\|_{C^3(\overline{B_1})}\Theta(C_1)\right)\right|^3} + \frac{\left\|D^2F\right\|_{L^\infty(B_{C_1})}}{\left|\tan\left(\|g\|_{C^3(\overline{B_1})}\Theta(C_1)\right)\right|}\right). \end{aligned}$$

The proof of the theorem follows the method in Theorem 9.5 [CC], which is the corresponding theorem for concave F. The difficulties of the proof of Theorem 3.2.1 occur since we do not know that u_{ee} is a subsolution of a linear operator for any direction e. In order to overcome the limitation, we utilize the fact that $e^{-Ka_{ij}D_{ij}u}$ is a subsolution of a linear operator (see Lemma 3.3.4) and a simple inequality, $x + 1 \leq e^x \leq \frac{e^a-1}{a}x + 1$ for $x \in [0, a]$.

proof of Theorem 3.2.1. Since F is uniformly elliptic, there exists $t \in \mathbb{R}$ such that $F(tI) = 0$ with $|t| \leq |F(0)|/\lambda$. Let $P(x) = \frac{t}{2}|x|^2$, $v = u - P$ and $G(\mathcal{M}) = F(\mathcal{M} + tI)$. Then $G(0) = 0$, $G(D^2v) = F(D^2u) = 0$ in B_1 and $\Sigma_G = \{\mathcal{M} \mid G(\mathcal{M}) = 0\} = \Sigma_F - tI$. Thus we may assume that $F(0) = 0$.

Let $v = u/\|g\|_{C^3(\overline{B_1})}$ and $G(\mathcal{M}) = \frac{1}{\|g\|_{C^3(\overline{B_1})}}F(\|g\|_{C^3(\overline{B_1})}\mathcal{M})$. Then G is uniformly elliptic with the same ellipticity constants as F and $G(D^2v) = 0$ in B_1. Since $\Sigma_G = \frac{\Sigma_F}{\|g\|_{C^3(\overline{B_1})}}$, it follows that $\omega_G = C(\|g\|_{C^3(\overline{B_1})})\omega_F$ and $\Theta_G = C(\|g\|_{C^3(\overline{B_1})})\Theta_F$. Thus we may assume that $\|g\|_{C^3(\overline{B_1})} \leq 1$.

By Proposition 9.1 of [CC], we obtain $u \in C^\infty(B_1)$ and by the proof of Theorem 9.5 of [CC], we know that $\|u\|_{L^\infty(B_1)}$, $\|\nabla u\|_{L^\infty(B_1)}$ and $\left\|D^2u\right\|_{L^\infty(\partial B_1)}$ are bounded by a universal constant C. Thus the remaining steps are to show boundedness of $\left\|D^2u\right\|_{L^\infty(B_1)}$ and $\left\|D^2u\right\|_{C^\alpha(\overline{B_1})}$.

1) $\left\|D^2u\right\|_{L^\infty(\partial B_1)} < C$ implies $\left\|D^2u\right\|_{L^\infty(B_1)} < C_1$.

Let $v = -u$ and $A = (a_{ij})$ be a symmetric $n\times n$ matrix such that $\lambda' I \leq A \leq \Lambda' I$. By Lemma 3.3.4, $e^{Ka_{ij}D_{ij}v}$ is a subsolution of a linear operator in B_1, for

$$\begin{aligned} K &= K(n,\lambda,\Lambda,\lambda',\Lambda',\omega(M_0),\Theta(M_0),\left\|D^2F\right\|_{L^\infty(B_{M_0})}) \\ &= C(n,\lambda,\Lambda,\lambda',\Lambda')\left(\frac{\left\|D^2F\right\|^2_{L^\infty(B_{M_0})}}{\omega(M_0)|\tan\Theta(M_0)|^3} + \frac{\left\|D^2F\right\|_{L^\infty(B_{M_0})}}{|\tan\Theta(M_0)|^2}\right), \end{aligned}$$

where $M_0 = \left\|D^2u\right\|_{L^\infty(B_1)}$.

Since we have $\left\|D_{ij}v\right\|_{L^\infty(\partial B_1)} < C$ and $\left\|e^{Ka_{ij}D_{ij}v}\right\|_{L^\infty(\partial B_1)} < e^{Kn\Lambda' C}$, by maximum principle, we know

$$\sup_{B_1} e^{Ka_{ij}D_{ij}v} = \sup_{\partial B_1} e^{Ka_{ij}D_{ij}v} < e^{Kn\Lambda' C}$$

and

$$\sup_{B_1} a_{ij}D_{ij}v < n\Lambda' C.$$

For a fixed unit vector $e = e_l$, we take a_{ij} such that

$$a_{ij} = \begin{cases} 1 & \text{if } i = j = l \\ \epsilon' & \text{if } i = j \text{ and } i \neq l \\ 0 & \text{if } i \neq j. \end{cases}$$

Since ϵ' is arbitrary,

$$\sup_{B_1} D_{ll}v < nC \quad \text{for any } l.$$

Moreover, we know that

$$\sup_{B_1} D_{ee}v < n^2C,$$

for any unit vector e. By Lemma 6.4 of [CC],

$$\left\|D^2u(x)\right\| = \left\|D^2v(x)\right\| \leq C_0 \sup_{|e|=1} D_{ee}v(x)^+ < C_0n^2C \quad \text{for all } x \in B_1.$$

2) Boundedness of $\left\|D^2u\right\|_{C^\alpha(\overline{B_1})}$.

By the proof of theorem 9.5 of [CC], we know that

$$\left|D_{ij}v(x^0) - D_{ij}v(x^1)\right| \leq C|x_1 - x_0|^\beta \quad \text{for all } x^0, x^1 \in \partial B_1,$$

for a universal constant $\beta \in (0,1)$. Fix $x^0, x^1 \in \partial B_1$ and a symmetric $n \times n$ matrix $A = (a_{ij})$ with $\lambda' I \leq A \leq \Lambda' I$. We may assume that $Ka_{ij}D_{ij}v(x^1) \geq Ka_{ij}D_{ij}v(x^0)$. $\left\|D^2v\right\|_{L^\infty(B_1)} < C_1$ implies $\left\|Ka_{ij}D_{ij}v\right\|_{L^\infty(B_1)} < Kn\Lambda' C_1$. Thus, by a simple inequality, $x + 1 \leq e^x \leq \frac{e^a - 1}{a}x + 1$ for $x \in [0, a]$, we have

$$\begin{aligned}
\left|e^{Ka_{ij}D_{ij}v(x^1)} - e^{Ka_{ij}D_{ij}v(x^0)}\right| &= e^{Ka_{ij}D_{ij}v(x^1)} - e^{Ka_{ij}D_{ij}v(x^0)} \\
&= e^{Ka_{ij}D_{ij}v(x^0)}\left(e^{Ka_{ij}D_{ij}v(x^1) - Ka_{ij}D_{ij}v(x^0)} - 1\right) \\
&\leq e^{Ka_{ij}D_{ij}v(x^0)}\frac{e^{2Kn\Lambda' C_1} - 1}{2Kn\Lambda' C_1}\left(Ka_{ij}D_{ij}v(x^1) - Ka_{ij}D_{ij}v(x^0)\right) \qquad (3.4) \\
&\leq e^{Kn\Lambda' C_1}\frac{e^{2Kn\Lambda' C_1} - 1}{2Kn\Lambda' C_1}Kn\Lambda'\left|D_{ii}v(x^1) - D_{ii}v(x^0)\right| \\
&\leq e^{Kn\Lambda' C_1}\frac{e^{2Kn\Lambda' C_1} - 1}{2Kn\Lambda' C_1}Kn\Lambda' C|x_1 - x_0|^\beta,
\end{aligned}$$

i.e., we obtain

$$\left|e^{Ka_{ij}D_{ij}v(x^1)} - e^{Ka_{ij}D_{ij}v(x^0)}\right| \leq C(K, \Lambda')|x_1 - x_0|^\beta \quad \text{for all } x^0, x^1 \in \partial B_1.$$

Since $e^{Ka_{ij}D_{ij}v}$ is a subsolution of a linear operator and $e^{Ka_{ij}D_{ij}v} \in C^\beta(\partial B_1)$, we obtain that

$$\left|e^{Ka_{ij}D_{ij}v(x)} - e^{Ka_{ij}D_{ij}v(x^0)}\right| \leq C(K, \Lambda')|x - x_0|^{\beta/2} \quad \forall x \in B_1, \forall x^0 \in \partial B_1,$$

by Proposition 4.12 of [CC].

Let $G(\mathcal{M}) = -F(-\mathcal{M})$. Then $G(D^2v) = 0$ in B_1 and $\Sigma' = \{\mathcal{M} \mid G(\mathcal{M}) = 0\}$ satisfies (E'3) and (E'4), since $\Sigma = \{\mathcal{M} \mid F(\mathcal{M}) = 0\}$ satisfies (E3) and (E4). Thus by [CY], we know the interior $C^{2,\gamma}$ regularity of v for some universal constant $\gamma \in (0,1)$. By the same method as in (3.4), $e^{Ka_{ij}D_{ij}v}$ also has interior $C^{2,\gamma}$ regularity. Thus, by using the same method as in Proposition 4.13 of [CC], we obtain $C^{2,\alpha}(\overline{B_1})$ boundedness of $e^{Ka_{ij}D_{ij}v}$ for $\alpha = \min(\gamma, \beta/2)$.

Fix $x, y \in \overline{B_1}$ and assume that $e^{Ka_{ij}D_{ij}v(x)} \geq e^{Ka_{ij}D_{ij}v(y)}$. Then we have

$$\begin{aligned} &e^{-Kn\Lambda' C_1}\left(Ka_{ij}D_{ij}v(x) - Ka_{ij}D_{ij}v(y)\right) \leq e^{-Kn\Lambda' C_1}\left(e^{Ka_{ij}D_{ij}v(x) - Ka_{ij}D_{ij}v(y)} - 1\right) \\ &\leq e^{Ka_{ij}D_{ij}v(y)}\left(e^{Ka_{ij}D_{ij}v(x) - Ka_{ij}D_{ij}v(y)} - 1\right) = e^{Ka_{ij}D_{ij}v(x)} - e^{Ka_{ij}D_{ij}v(y)}. \end{aligned}$$

So,

$$\begin{aligned} \left|a_{ij}D_{ij}v(x) - a_{ij}D_{ij}v(y)\right| &\leq \frac{e^{Kn\Lambda' C_1}}{K}\left|e^{Ka_{ij}D_{ij}v(x)} - e^{Ka_{ij}D_{ij}v(y)}\right| \\ &\leq C(K, \Lambda')|x - y|^{\alpha}, \end{aligned}$$

i.e, we obtain that $a_{ij}D_{ij}v \in C^{0,\alpha}(\overline{B_1})$. We may assume that $a_{ij}D_{ij}v = \Delta v$. Then by Theorem 6.6 of [GT], we have that

$$\begin{aligned} \|u\|_{C^{2,\alpha}(\overline{B_1})} = \|v\|_{C^{2,\alpha}(\overline{B_1})} &\leq C\left(\|v\|_{L^\infty(\overline{B_1})} + \|g\|_{C^3(\overline{B_1})} + \|\Delta v\|_{C^\alpha(\overline{B_1})}\right) \\ &\leq C + C(K), \end{aligned}$$

where

$$\begin{aligned} K &= K\left(n, \lambda, \Lambda, \omega(C_1), \Theta(C_1), \left\|D^2F\right\|_{L^\infty(B_{C_1})}\right) \\ &= C(n, \lambda, \Lambda)\left(\frac{\left\|D^2F\right\|^2_{L^\infty(B_{C_1})}}{\omega(C_1)|\tan\Theta(C_1)|^3} + \frac{\left\|D^2F\right\|_{L^\infty(B_{C_1})}}{|\tan\Theta(C_1)|}\right), \end{aligned}$$

since we know that

$$\left\|D^2u\right\|_{L^\infty(B_1)} \leq C_1.$$

$\square$

Remark. In the proof of the next theorem, we are going to use the method of continuity, see Proof of Theorem 9.7 of [CC]. Thus, we need to have a priori $C^{2,\alpha}$ estimate up to the boundary for

$$\begin{cases} tF(D^2u) + (1-t)\Delta u = 0 & \text{in } B_1, \\ u = g & \text{on } \partial B_1, \end{cases} \tag{3.5}$$

which does not depend on t. Let $u^t \in C^{2,\alpha}(B_1)$ be a solution of (3.5) for $t \in [0, 1]$ and suppose that $\Sigma_t = \{\mathcal{M} \,|\, tF(\mathcal{M}) + (1-t)tr(\mathcal{M}) = 0\}$ satisfies $(E4)$ and $(E5)$, for any $t \in [0, 1]$ with ω_t and Θ_t. By Theorem 3.2.1,

$$\left\|D^2u^t\right\|_{L^\infty(B_1)} \leq C_1,$$

for the universal constant C_1 and

$$\left\|u^t\right\|_{C^{2,\alpha}(\overline{B_1})} \le C(K_t)\left(\|g\|_{C^3(\overline{B_1})} + |F(0)|\right),$$

for a universal constant $\alpha \in (0,1)$, where

$$\begin{aligned}K_t &= K_t\left(n, \lambda, \Lambda, \omega_t(C_1), \Theta_t(C_1), \left\|D^2F\right\|_{L^\infty(B_{C_1})}, \|g\|_{C^3(\overline{B_1})}\right)\\ &= C\left(n, \lambda, \Lambda, \|g\|_{C^3}\right)\left(\frac{t^2\left\|D^2F\right\|^2_{L^\infty(B_{C_1})}}{\omega_t(C_1)\left|\tan\left(\|g\|_{C^3(\overline{B_1})}\Theta_t(C_1)\right)\right|^3} + \frac{t\left\|D^2F\right\|_{L^\infty(B_{C_1})}}{\left|\tan\left(\|g\|_{C^3(\overline{B_1})}\Theta_t(C_1)\right)\right|}\right).\end{aligned}$$

Hence, we need a condition such that K_t is uniformly bounded under the condition. Therefore, we assume conditions for the decay rate of ω_t and Θ_t. More Specifically, we assume that $\omega_t(C) \le h(t) \in o(t^{1/2})$ and $\Theta_t(C) \le k(t) \in o(t)$.

Theorem 3.2.2. *Let $F(\mathcal{M}) \in C^\infty(\mathbb{R}^{n\times n})$ satisfy $(E2)$ and $(E3)$,*

$$\Sigma_t = \{\mathcal{M} \mid tF(\mathcal{M}) + (1-t)tr(\mathcal{M}) = 0\}$$

satisfy $(E4)$ and $(E5)$ for all $t \in [0,1]$ with $\omega_t(C_1)$ and $\Theta_t(C_1)$, where C_1 is the universal constant in Theorem 3.2.1. Assume that there are functions $h, k : \mathbb{R} \to \mathbb{R}$ such that

$$\omega_t(C_1) \le h(t) \in o(t^{1/2}) \quad \text{and} \quad \Theta_t(C_1) \le k(t) \in o(t).$$

Then for $g \in C(\partial B_1)$, there exists a unique viscosity solution $u \in C^\infty(B_1)$ of

$$\begin{cases} F(D^2u) = 0 & \text{in } B_1,\\ u = g & \text{on } \partial B_1\end{cases} \tag{3.6}$$

and

$$\begin{aligned}\|u\|_{C^{1,1}(B_{1/2})} &\le C(n, \lambda, \Lambda)(\|u\|_{L^\infty(B_1)} + |F(0)| + \chi),\\ \left\|D^2u\right\|_{C^\alpha(B_{1/2})} &\le C(n, \lambda, \Lambda, \alpha, \omega(C_1), \Theta(C_1), \left\|D^2F\right\|_{L^\infty(B_{C_1})}, C_1).\end{aligned}$$

Proof. First, we assume that $g \in C^\infty(\partial B)$. Then, by Theorem 3.2.1 and Remark 3.2, we have

$$\left\|D^2u^t\right\|_{L^\infty(B_1)} \le C_1,$$

and

$$\left\|u^t\right\|_{C^{2,\alpha}(\overline{B_1})} \le C(n, \lambda, \left\|D^2F\right\|_{L^\infty(B_{C_1})}, \|g\|_{C^3(\overline{B_1})})\left(\|g\|_{C^3(\overline{B_1})} + |F(0)|\right),$$

where u^t is a solution of (3.5) for $t \in [0, 1]$.

Thus by the method of continuity, we know that there is a solution $u \in C^{2,\alpha}(\overline{B_1})$ of (3.6), see proof of theorem 9.7 in [CC]. Since $F(\mathcal{M}) \in C^\infty(\mathbb{R}^{n\times n})$, $u \in C^\infty(B_1)$, by Proposition 9.1 of [CC]. Therefore, Theorem 1, Theorem 2 in [CY] for $v = -u$ implies

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C(n, \lambda, \Lambda)(\|u\|_{L^\infty(B_1)} + |F(0)| + \chi),$$
$$\left\|D^2 u\right\|_{C^\alpha(B_{1/2})} \leq C(n, \lambda, \Lambda, \alpha, \omega(C_1), \Theta(C_1), \left\|D^2 F\right\|_{L^\infty(B_{C_1})}, C_1).$$

By using the method in the proof of Proposition 9.8 of [CC], we have the same conclusion for $g \in C(\partial B_1)$. □

3.3 Regularity of the Free Boundary

In this subsection, we are going to prove the main theorem of this section, Theorem 3.1.1. The main result in Subsection 3.3.2 is that $u \in P_\infty(M)$ is convex and a half-space solution under the thickness assumption for u. In Subsection 3.3.3 and 3.3.4, we discuss the directional monotonicity and the proof of Theorem 3.1.1. The methods in the subsections are almost the same as that for linear operator and convex fully nonlinear operator in [Fri], [Lee98], [PSU], [FS14], [IM16a] and [LPS].

3.3.1 General Properties

Lemma 3.3.1 (Non-degeneracy)**.** *Let $u \in P_1(M)$ and $v = u - \phi$. If $-F(D^2\phi) \geq c > 0$ in B_1, then*

$$\sup_{y\in B_r(x)} v(y) \geq v(x) + c\frac{r^2}{2\Lambda}, \quad x \in \overline{\Omega(v)} = \overline{\{v > 0\}} \cap B_1,$$

for any $B_r(x) \subset B_1$.

Proof. Since $u \in P_1(M)$, $v = u - \phi$ is a solution of

$$\tilde{F}(D^2 v, x) = f(x)\chi_{\{v>0\}}, \quad v \geq 0 \quad \text{in } B_r,$$

where $\tilde{F}(\mathcal{M}, x) = F(\mathcal{M} + D^2\phi) - F(D^2\phi)$ and $f(x) = -F(D^2\phi) \geq c > 0$ in B_1. Let $x^0 \in \Omega(v) = \{v > 0\}$ and define the auxiliary function

$$w(x) = v(x) - v(x^0) - c\frac{|x - x^0|^2}{2\Lambda}.$$

Then $F(D^2w, x) = F\left(D^2v - (c/\Lambda)I, x\right) \geq F(D^2v, x) - c = f(x) - c \geq 0$ in $\Omega(v) \cap B_r(x^0)$. By the maximum principle on $\Omega(v) \cap B_r(x^0)$ and $w(x^0) = 0$,

$$\sup_{\partial\left(\Omega(v)\cap B_r(x^0)\right)} w \geq 0.$$

Since $v = 0$ on $\partial\Omega(v) \cap B_1$, we know that $w(x) = -v(x^0) - c\left(|x - x^0|^2\right)/2\Lambda < 0$ on $\partial\Omega(v) \cap B_r(x^0)$. Thus we have that

$$\sup_{\partial B_r(x^0)\cap\Omega(v)} w \geq 0 \quad \text{and} \quad \sup_{\partial B_r(x^0)\cap\Omega(v)} v \geq v(x^0) + \frac{cr^2}{2\Lambda}.$$

Since v is a subsolution of $\tilde{F}(D^2v, x) = 0$ in B_1, we know that $\sup_{B_\rho(x^0)} v = \sup_{\partial B_\rho(x^0)} v$ and

$$\sup_{B_r(x^0)} v \geq \sup_{\partial B_r(x^0)\cap\Omega(v)} v \geq v(x) + c\frac{\rho^2}{2\Lambda}.$$

Let $x^0 \in \partial\Omega(v)$ and $x^i \in \Omega(v)$ be a sequence of points converging to x as $i \to \infty$. By passing to the limit as i goes to ∞, we have the desired inequalities for $x^0 \in \overline{\Omega(v)} \cap B_1$. $\square$

Definition 3.3.1. We say a mesurable set $A \subset \mathbb{R}^n$ is porous with a porosity $0 < \delta < 1$ if for any point x and $B_r(x)$ there is a ball $B = B_{\delta r}(y) \subset B_r(x)$ such that $B \cap A = \emptyset$. We say that A is locally porous in an open set D if $A \cap K$ is porous, for any compact set K in D.

For x in a porous set A,

$$\frac{\int_{B_r(x)} \chi_A dx}{|B_r(x)|} = \frac{|B_r(x) \cap A|}{|B_r(x)|} \leq 1 - \delta^n < 1,$$

for any $r > 0$. Moreover, we know that $\lim_{r\to 0} \int_{B_r(x)} \chi_A dx/|B_r(x)| = 1$ for almost all $x \in A$, since χ_A is a locally L^1 function. Hence, we conclude that if A is porous then A is Lebesgue measure zero.

Lemma 3.3.2. *Let $u \in P_1(M)$. If $-F(D^2\phi) \geq c > 0$ in B_1, then $\Gamma(u)$ is locally porous in B_1. Then, $\Gamma(u)$ has a Lebesgue measure zero.*

Proof. Let K be a compact set in B_1, $x_0 \in \Gamma(u) \cap K$ and $B_r(x_0) \subset K$. By the non-degeneracy (Lemma 3.3.1), there is a point $y \in \partial B_{r/2}(x_0)$ such that

$v(y) \geq c\frac{r^2}{4\Lambda}$. Let $d = dist(y, \Gamma(u)) = dist(y, \Gamma(v))$ and take $z \in \Gamma(u) \cap \partial B_d(y)$. Since $\left\|D^2 v\right\|_{L^\infty(B_1)} \leq M_0 := M + \left\|D^2\phi\right\|_{L^\infty(B_1)}$, $u(y) \leq \frac{M_0}{2}d^2$. Then, we conclude that

$$c\frac{r^2}{4\Lambda} \leq v(y) \leq \frac{M_0}{2}d^2 \text{ and } \sqrt{\frac{c}{2\Lambda M_0}}r \leq d.$$

We may assume that $\delta := \sqrt{\frac{c}{2\Lambda M_0}} < 1/2$. Then, a ball $B_{\delta r}(y)$ is in $B_r(x_0)$ and $B_{\delta r}(y) \cap \Gamma(u) = \emptyset$. This implies the local porosity of $\Gamma(u)$. Consequently, $\Gamma(u) = \Gamma(v)$ has a Lebesgue measure zero. □

Lemma 3.3.3. *Let $u \in P_1(M)$ with the obstacle function ϕ such that $-F(D^2\phi) \geq c > 0$ in B_1 and assume that $F(\mathcal{M}) \in C^\infty(\mathbb{R}^{n\times n})$ and Σ_t satisfy conditions in Theorem 3.2.2. Then any blowup v_0 of $v = u - \phi$ at 0 is a solution of*

$$\tilde{F}(D^2 v_0, 0) = f(0)\chi_{\{v_0>0\}}, \quad v_0 \geq 0 \quad a.e.\ in\ \mathbb{R}^n,$$

for $\tilde{F}(\mathcal{M}, x) = F(\mathcal{M} + D^2\phi) - F(D^2\phi)$, $f(x) = -F(D^2\phi)$. Moreover, $v_0 \in P_\infty(M)$ and $v_0 \in C^{2,\beta}(\{v_0 > 0\})$, for some $\beta \in (0, 1)$.

Proof. Since $u \in P_1(M)$, $v = u - \phi$ is a solution of

$$\tilde{F}(D^2 v, x) = f(x)\chi_{\{v>0\}}, \quad v \geq 0 \quad \text{in } B_1,$$

where $\tilde{F}(\mathcal{M}, x) = F(\mathcal{M} + D^2\phi) - F(D^2\phi)$ and $\tilde{F}(\mathcal{M}, x), f(x) = -F(D^2\phi) \in C^{0,1}(B_1)$ with respect to x. Let v_{r_i} be a sequence of the rescaling functions converging to a blowup v_0. Then the rescaling v_{r_i} is solution of

$$\tilde{F}(D^2 v_{r_i}(x), r_i x) = f(r_i x)\chi_{\Omega(v_{r_i})} \quad \text{in } B_{1/r_i},$$

where $\Omega(v_{r_i}) := \{v_{r_i} > 0\}$.

Let $x \in \{v_0 > 0\}$. Then, by $C^{1,\alpha}_{loc}$ convergence of v_{r_i} to v_0, we know that there exist $\delta > 0$ and i_0 such that $B_\delta(x) \subset \Omega(v_{r_i})$ for all $i \geq i_0$. Then by the definition of $\tilde{F}$ and v, we know that

$$F(D^2 u_{r_i}(y)) = F(D^2 u(r_i y)) = 0 \quad \text{in } B_\delta(x),$$

where u_{r_i} is the rescaling functions of u at 0. By Theorem 3.2.2, we may assume strong convergence of u_{r_i}, v_{r_i} to u_0, v_0, respectively, in $C^{2,\beta}(B_\delta(x))$ for some $0 < \beta < \alpha$. Thus we have that

$$\begin{aligned}\tilde{F}(D^2 v_0(x), 0) = \lim_{i\to\infty} \tilde{F}(D^2 v_{r_i}(x), r_i^2 x) = \lim_{i\to\infty} f(r_i^2 x)\\ = f(0) > 0,\end{aligned}$$

$|D^2 v_0(x)| \leq M$ and $v_0 \in C^{2,\beta}(\Omega(v_0))$. Since $\tilde{F}(0,0) = 0$, $\tilde{F}(D^2 v_0, 0) = 0$ a.e. on $\{v_0 = 0\}$. Therefore, v_0 is a solution of

$$\tilde{F}(D^2 v_0, 0) = f(0)\chi_{\{v_0>0\}} \quad \text{a.e. in } \mathbb{R}^n.$$

Moreover, we obtain $0 \in \Gamma(v_0)$, by using the non-degeneracy, see e.g. the proof of Proposition 3.17 (iv) of [PSU]. Therefore, for the fully nonlinear operator $\tilde{F}(\mathcal{M}, 0)/f(0)$, v_0 is in $P_\infty(M)$ and $v_0 \in C^{2,\beta}(\Omega(v_0))$. □

3.3.2 Convexity of Global Solutions $u \in P_\infty(M)$

The first step of the proof for Theorem 3.1.1 is obtaining the convexity of $u \in P_\infty(M)$. If u is a solution of $F(D^2u) = a$, for some $a \in \mathbb{R}$ and a convex operator F, then it is easily obtained that u_{ee} ($e \in \mathbb{R}^n$, $|e| = 1$) is a supersolution of a linear elliptic operator. The result is used in the proof of the convexity of blowup (see [Lee98, FS14]). However, if u is a solution of $F(D^2u) = a$ under the assumptions for the level set $\Sigma = \{F(\mathcal{M}) = a\}$, (E4) and (E5), u_{ee} is no longer a supersolution of a linear operator. On the other hand, by modifying results in [CY], we know that $-e^{-Ka_{ij}D_{ij}u}$ is a supersolution of a linear operator, where $A = (a_{ij})$ is a symmetric $n \times n$ matrix with $\lambda' I \leq A \leq \Lambda' I$ for positive constants $0 < \lambda' \leq \Lambda' < \infty$ under the assumptions (E4) and (E5), see Lemma 3.3.4. Therefore, we have the convexity of $u \in P_\infty(M)$ and moreover we know that $u \in P_\infty(M)$ is a half-space solution, by using the method in [LS01, FS14, IM16a].

Lemma 3.3.4. *Let u be a smooth solution of $F(D^2u) = a$ in B_1 for some $a \in \mathbb{R}$ and assume that $F \in C^\infty(\mathbb{R}^{n\times n})$ satisfies (E2) and $\Sigma = \{F(\mathcal{M}) = a\}$ satisfies (E4) and (E5) with $\left\|D^2u\right\|_{L^\infty(B_1)} < M_0$ for some positive constant M_0. Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix such that $\lambda' I \leq A \leq \Lambda' I$. Then we have*

$$Le^{-Ka_{ij}D_{ij}u} = F_{ij}(D^2u)D_{ij}e^{-Ka_{ij}D_{ij}u} \geq 0 \quad \text{in } B_1,$$

for a sufficiently large $K = K(n, \lambda, \lambda', \Lambda', \omega(M_0), \Theta(M_0), \left\|D^2F\right\|_{L^\infty(B_{M_0})})$.

Proof. For a symmetric matrix $A = (a_{ij})$ with $\lambda' I \leq A \leq \Lambda' I$, there is a diagonal matrix B such that $(B)_{\alpha\beta} = b_\alpha \delta_{\alpha\beta}$ and $\lambda' \leq b_\alpha \leq \Lambda'$ such that $tr(AD^2u) = tr(BD^2u)$. By differentiating $F(D^2u) = 0$ with respect to a unit vector α, we have $F_{ij}D_{ij}u_\alpha = 0$ and $F_{ij}D_{ij}u_{\alpha\alpha} + F_{ij,kl}D_{ij}u_\alpha D_{kl}u_\alpha = 0$. Thus,

$$\begin{aligned}
Le^{-Ktr(AD^2u)} &= Le^{-Ktr(BD^2u)} \\
&= \sum_{\alpha=1}^{n}\Big(-Ke^{-Kb_\alpha D_{\alpha\alpha}u}F_{ij}D_{ij}(b_\alpha D_{\alpha\alpha}u) \\
&\qquad\qquad +K^2e^{-Kb_\alpha D_{\alpha\alpha}u}F_{ij}D_i(b_\alpha D_{\alpha\alpha}u)D_j(b_\alpha D_{\alpha\alpha}u)\Big) \\
&= \sum_{\alpha=1}^{n}Ke^{-Kb_\alpha D_{\alpha\alpha}u}\Big[F_{ij,kl}b_\alpha D_{ij}u_\alpha D_{kl}u_\alpha \\
&\qquad\qquad +KF_{ij}D_i(b_\alpha D_{\alpha\alpha}u)D_j(b_\alpha D_{\alpha\alpha}u)\Big].
\end{aligned}$$

Suppose $F_{ij,kl}(D^2u(x)) \geq 0$ for $x \in B_1$. Since

$$\sum_{\alpha=1}^{n}KF_{ij}D_i(b_\alpha D_{\alpha\alpha}u)D_j(b_\alpha D_{\alpha\alpha}u) \geq \frac{K\lambda(\lambda')^2}{n}|D(\Delta u)|^2,$$

we have $Le^{-Ktr(AD^2u)} \geq 0$.

Suppose the other case, $F_{ij,kl}(D^2u(x)) \ngeq 0$, for $x \in B_1$. We denote

$$\tilde{\mathrm{I}} = \sum_{\alpha=1}^{n}F_{ij,kl}b_\alpha D_{ij}u_\alpha D_{kl}u_\alpha \quad \text{and} \quad \tilde{\mathrm{II}} = \sum_{\alpha=1}^{n}KF_{ij}D_i(b_\alpha D_{\alpha\alpha}u)D_j(b_\alpha D_{\alpha\alpha}u).$$

By the uniform ellipticity of F with ellipticity constants λ, Λ, we know that the directional derivative of F in the direction I, $\partial F/\partial I$ is greater than λ. Then by the implicit function theorem, there is a function $g :< I >^\perp \subset \mathbb{R}^n \to \mathbb{R}$ such that Σ is represented by the graph of g. More precisely, we choose a unitary $n^2 \times n^2$ matrix $\mathcal{A}$ such that $p_{nn} = (m_{11} + m_{22} + ... + m_{nn})/n$, where $\mathcal{P} = (p_{ij})$, $\mathcal{M} = (m_{ij})$ and $\mathcal{AP} = \mathcal{M}$. Define $\tilde{F}(\mathcal{P}) := F(\mathcal{AP}) = F(\mathcal{M})$ and $\bar{\mathcal{B}}(x) = (\bar{b}_{ij}(x)) := \mathcal{A}^{-1}D^2u(x)$. Then, $\tilde{F}(\bar{\mathcal{B}}(x)) = F(D^2u(x)) = a$ and

$$\partial_{p_{nn}}\tilde{F}(\mathcal{P}) = \partial_{m_{ij}}F(\mathcal{M})\frac{\partial_{m_{ij}}}{\partial_{p_{nn}}} = \partial_{I/n}F(\mathcal{M}) \neq 0.$$

Thus, $\Sigma = \{\mathcal{M} \in \mathbb{R}^{n\times n} \,|\, F(\mathcal{M}) = a\} = \{\mathcal{P} \in \mathbb{R}^{n\times n} \,|\, \tilde{F}(\mathcal{P}) = a\}$ can be represented as a graph along the direction p_{nn} in $\mathbb{R}^{n\times n}$. By implicit function theorem, there exists a function $g : \mathbb{R}^{n\times n-1} \to \mathbb{R}$, such that $\tilde{F}(p', g(p')) = 0$, where $p' = (p_{11}, ..., p_{nn-1})$.

By differentiating $\tilde{F}(p', g(p')) = 0$, we have

$$\tilde{F}_{ij} + \tilde{F}_{nn} g_{ij} = 0, \quad ij \neq nn. \tag{3.7}$$

Since $\tilde{F}(\bar{b}_{11}(x), ..., \bar{b}_{nn-1}(x), g(\bar{b}_{11}(x), ..., \bar{b}_{nn-1}(x))) = 0$ and $\bar{\mathbb{B}}(x) = \mathcal{A}^{-1} D^2 u(x)$, we know

$$\bar{b}_{nn}(x) = \Delta u(x) = g(\bar{b}_{11}(x), ..., \bar{b}_{nn-1}(x))$$

and by differentiating, we have

$$D_\alpha \bar{b}_{nn} = g_{ij}(\bar{b}_{11}(x), ..., \bar{b}_{nn-1}(x)) D_\alpha \bar{b}_{ij}(x), \quad \text{for any } ij \neq nn \tag{3.8}$$

and

$$D_{\alpha\alpha} \bar{b}_{nn} = g_{ij,kl} D_\alpha \bar{b}_{ij}(x) D_\alpha \bar{b}_{kl}(x) + g_{ij} D_{\alpha\alpha} \bar{b}_{ij}(x), \quad \text{for any } ij, kl \neq nn.$$

Thus,

$$\begin{aligned}
\tilde{F}_{nn} \cdot \left(g_{ij,kl} D_\alpha \bar{b}_{ij}(x) D_\alpha \bar{b}_{kl}(x) \right) =& \tilde{F}_{nn} \cdot \left(-g_{ij} D_{\alpha\alpha} \bar{b}_{ij} + D_{\alpha\alpha} \bar{b}_{nn} \right) \text{ for any } ij, kl \neq nn \\
=& \tilde{F}_{ij} D_{\alpha\alpha} \bar{b}_{ij} + \tilde{F}_{nn} D_{\alpha\alpha} \bar{b}_{nn} \quad \text{for any } ij \neq nn \\
=& \tilde{F}_{ij} D_{\alpha\alpha} \bar{b}_{ij} \quad \text{for any } ij \\
=& -\tilde{F}_{ij,kl} D_\alpha \bar{b}_{ij} D_\alpha \bar{b}_{kl} = -{}^t \left(D_\alpha \bar{\mathbb{B}}(x) \right) D^2 \tilde{F} D_\alpha \bar{\mathbb{B}}(x) \\
=& -{}^t \left(\mathcal{A}^{-1} D^2 u_\alpha(x) \right) {}^t \mathcal{A} D^2 F \mathcal{A} \mathcal{A}^{-1} D^2 u_\alpha(x) \\
=& -{}^t D^2 u_\alpha(x) D^2 F D^2 u_\alpha(x).
\end{aligned}$$

By equation (3.8), $D_\alpha \bar{b}_{ij}$ is decomposed by $D_\alpha \bar{b}_{ij} = T_\alpha + [D_\alpha(\bar{b}_{nn})](Dg/|Dg|^2) + D_\alpha \bar{b}_{nn} \epsilon_{nn}$ with $T_\alpha \perp \epsilon_{nn}, T_\alpha \perp Dg, Dg = (g_{ij})$,where $\epsilon_{nn} = e_n \otimes e_n$.

By the definition of g and condition (E4),

$$
\begin{aligned}
-\tilde{I} &= \sum_{\alpha=1}^{n} \tilde{F}_{nn} b_\alpha g_{ij,kl} D_\alpha \bar{b}_{ij} D_\alpha \bar{b}_{ij} \\
&= \sum_{\alpha=1}^{n} \tilde{F}_{nn} b_\alpha D^2 g\left(T_\alpha + D_\alpha(\Delta u)\frac{Dg}{|Dg|^2}, T_\alpha + D_\alpha(\Delta u)\frac{Dg}{|Dg|^2}\right) \\
&= \sum_{\alpha=1}^{n} \tilde{F}_{nn} b_\alpha \left[D^2 g(T_\alpha, T_\alpha) + 2D^2 g\left(T_\alpha, D_\alpha(\Delta u)\frac{Dg}{|Dg|^2}\right)\right. \\
&\qquad \left. + D^2 g\left(D_\alpha(\Delta u)\frac{Dg}{|Dg|^2}, D_\alpha(\Delta u)\frac{Dg}{|Dg|^2}\right)\right] \\
&= \sum_{\alpha=1}^{n} \tilde{F}_{nn} b_\alpha \left[|Dg|\mathrm{II}(T_\alpha, T_\alpha) + 2D^2 g\left(T_\alpha, D_\alpha(\Delta u)\frac{Dg}{|Dg|^2}\right) + \frac{|D(\Delta u)|^2}{|Dg|^2} g_{Dg/|Dg|, Dg/|Dg|}\right] \\
&\le \sum_{\alpha=1}^{n} \tilde{F}_{nn} b_\alpha \left[|Dg|\mathrm{II}(T_\alpha, T_\alpha) + 2\left\|D^2 g\right\|_{L^\infty(B_{M_0})} |T_\alpha||D_\alpha(\Delta u)|\frac{1}{|Dg|}\right. \\
&\qquad \left. + \frac{\left\|D^2 g\right\|_{L^\infty(B_{M_0})}}{|Dg|^2}|D(\Delta u)|^2\right] \\
&\le \sum_{\alpha=1}^{n} \tilde{F}_{nn} b_\alpha \left[-|Dg|\omega(M_0)|T_\alpha|^2 + \omega(M_0)|Dg||T_\alpha|^2 + \frac{\left\|D^2 g\right\|^2_{L^\infty(B_{M_0})}}{\omega(M_0)|Dg|^3}|D(\Delta u)|^2\right. \\
&\qquad \left. + \frac{\left\|D^2 g\right\|_{L^\infty(B_{M_0})}}{|Dg|^2}|D(\Delta u)|^2\right] \\
&\le \Lambda n \Lambda' \left[\frac{\left\|D^2 g\right\|^2_{L^\infty(B_{M_0})}}{\omega(M_0)|Dg|^3} + \frac{\left\|D^2 g\right\|_{L^\infty(B_{M_0})}}{|Dg|^2}\right]|D(\Delta u)|^2.
\end{aligned}
$$

By equation (3.7), we have $D\tilde{F} = (\tilde{F}_{11}, ..., \tilde{F}_{nn-1}, \tilde{F}_{nn}) = \tilde{F}_{nn}(-g_{11}, ..., -g_{ij}, 1)$. Then, $D\tilde{F} \cdot (0, .., 0, 1) = \frac{\tilde{F}_{nn}}{|\tilde{F}_{nn}|\sqrt{g_{ij}^2+1}} = \cos\theta$ and $|Dg| = |\tan\theta|$, where θ is the angle between the normal $\tilde{F}_{ij}$ to the level set Σ and $(0, ..., 0, 1)$. By the condition (E5) and the fact that

$$
\left\|D^2 g\right\|_{L^\infty(B_{M_0})} \le C(n, \lambda, \Lambda)\left\|D^2 F\right\|_{L^\infty(B_{M_0})},
$$

we have

$$\tilde{\mathrm{I}} \geq -\Lambda n\Lambda' C(n,\lambda,\Lambda)\left(\frac{\left\|D^2F\right\|^2_{L^\infty(B_E)}}{\omega(M_0)|\tan\Theta(M_0)|^3} + \frac{\left\|D^2F\right\|_{L^\infty(B_{M_0})}}{|\tan\Theta(M_0)|^2}\right)|D(\Delta u)|^2.$$

For a sufficiently large K such that

$$\begin{aligned}K &= K(n,\lambda,\lambda',\Lambda',\omega,\Theta,\left\|D^2F\right\|_{L^\infty(B_{M_0})})\\ &\geq \Lambda n\Lambda' C(n,\lambda,\Lambda)\left(\frac{\left\|D^2F\right\|^2_{L^\infty(B_{M_0})}}{\omega(M_0)|\tan\Theta(M_0)|^3} + \frac{\left\|D^2F\right\|_{L^\infty(B_{M_0})}}{|\tan\Theta(M_0)|^2}\right)\frac{n}{\lambda(\lambda')^2},\end{aligned}$$

we obtain

$$\begin{aligned}&\tilde{\mathrm{I}} + \tilde{\mathrm{II}} \geq\\ &\left[-\Lambda n\Lambda' C(n,\lambda,\Lambda)\left(\frac{\left\|D^2F\right\|^2_{L^\infty(B_{M_0})}}{\omega(M_0)|\tan\Theta(M_0)|^3} + \frac{\left\|D^2F\right\|_{L^\infty(B_{M_0})}}{|\tan\Theta(M_0)|^2}\right) + \frac{K\lambda(\lambda')^2}{n}\right]|D(\Delta u)|^2\\ &\geq 0.\end{aligned}$$

Therefore,

$$Le^{-Ktr(AD^2u)} = F_{ij}(D^2u)D_{ij}e^{-Ka_{ij}D_{ij}u} \geq 0,$$

for $K = K(n,\lambda,\Lambda,\lambda',\Lambda',\omega(M_0),\Theta(M_0),\left\|D^2F\right\|_{L^\infty(B_{M_0})})$. □

Proposition 3.3.5. *Let $u \in P_\infty(M)$ and assume that u is in $C^{2,\alpha}(\Omega(u))$, F satisfies $(E1)$-$(E3)$ and $\Sigma = \{\mathcal{M}\,|\,F(\mathcal{M}) = 1\}$ satisfies $(E4)$ and $(E5)$. Then u is a convex function in $\mathbb{R}^n$.*

Proof. Let $A = (a_{ij})$ be a symmetric $n\times n$ matrix such that $\lambda' I \leq A \leq \Lambda' I$. Suppose that

$$-m := \inf_{x\in\Omega(u)} a_{ij}D_{ij}u(x) < 0,$$

and take a sequence $x^k \in \Omega(u)$ such that

$$\lim_{k\to\infty} a_{ij}D_{ij}u(x^k) = -m.$$

Define rescaling functions u_k such that

$$u_k(x) = \frac{u(d_kx + x^k)}{d_k^2},$$

where $d_k = dist(x^k, \partial\Omega(u))$. Then B_1 is contained in $\{u_k > 0\}$ and ∂B_1 contains at least one point on $\partial\{u_k > 0\}$. By the uniform boundedness of u_k, which is $|D^2 u_k| \leq M$, we have

$$|u_k(x)| \leq \frac{M}{2}(|x| + 1)^2.$$

Thus, there is a global solution $\tilde{u} \in P_\infty(M)$ such that u_k converges to $\tilde{u}$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$, up to subsequence.

By Proposition 9.1 of [CC] and $u_k \in C^{2,\alpha}(B_1)$, we know that $u_k \in C^\infty(B_1)$. Thus, by Theorem 3.2.1 for $F(D^2 u_k) = 1$ on B_1, we may assume strong convergence of u_k to $\tilde{u}$ in $C^{2,\beta}(B_1)$ for some $0 < \beta < \alpha$. Hence, we have $F(D^2\tilde{u}) = 1$ on B_1. By Proposition 9.1 of [CC] again, we obtain $\tilde{u} \in C^\infty(B_1)$. Moreover, we know that ∂B_1 contains at least one free boundary point x^0 of $\tilde{u}$.

Set $\tilde{\Omega}(\tilde{u}) = B_1 \cup \{\tilde{u} > 0\}$ and take $x \in \{\tilde{u} > 0\}$. Then, by $C^{1,\alpha}_{loc}$ convergence of u_k to $\tilde{u}$, we know that there exist $\delta > 0$ and i_0 such that $B_\delta(x) \subset \Omega(u_k)$ for all $i \geq i_0$. By the same argument as the previous paragraph, we have $\tilde{u} \in C^\infty(\tilde{\Omega}(\tilde{u}))$. Furthermore, we obtain

$$F(D^2\tilde{u}) = 1, \quad a_{ij}D_{ij}\tilde{u} \geq -m \quad \text{in } \tilde{\Omega}(\tilde{u}), \quad a_{ij}D_{ij}\tilde{u}(0) = -m$$

and, by Lemma 3.3.4, $F_{ij}(D^2\tilde{u})\partial_{ij}(-e^{-Ka_{ij}D_{ij}\tilde{u}}) \leq 0$ on $\tilde{\Omega}(\tilde{u})$. Therefore, by the strong maximum principle, we have

$$a_{ij}D_{ij}\tilde{u} \equiv -m \quad \text{in } \tilde{\Omega}(\tilde{u}).$$

Thus by the strong maximum principle and $\tilde{u} \geq 0$ in $\mathbb{R}^n$, we know $\tilde{u} > 0$ on $\tilde{\Omega}(\tilde{u})$. Take $x \in \tilde{\Omega}(\tilde{u})$ and consider the ball $B := B_{r_0}(x)$ with $r_0 = dist(x, \partial\tilde{\Omega}(\tilde{u}))$. Let $y \in \partial\tilde{\Omega}(\tilde{u}) \cap \partial B_{r_0}$. Then, by Hopf's Lemma, $\frac{\partial\tilde{u}}{\partial\nu}(y) < 0$, where ν is the outer unit normal to B at y. Since $\tilde{u}$ has its minimum 0 on $\tilde{\Gamma}(\tilde{u})$, $D\tilde{u} \equiv 0$ on $\Gamma(u)$. Therefore, we arrive at a contradiction, i.e., we know that $a_{ij}D_{ij}u \geq 0$ in $\mathbb{R}^n$.

For a fixed unit vector $e = e_l$, we take a_{ij} such that

$$a_{ij} = \begin{cases} 1 & \text{if } i = j = l \\ \epsilon' & \text{if } i = j \text{ and } i \neq l \\ 0 & \text{if } i \neq j. \end{cases}$$

Since ϵ' is arbitrary, we know that $D_{ll}u \geq 0$ in $\mathbb{R}^n$ for any direction e_k. Thus u is convex in $\mathbb{R}^n$. □

In order to obtain that the global solution of the obstacle problem with the thickness assumption is half-space solution, (Proposition 3.3.7), we need to find a barrier function v that satisfies:

(i) v is a subsolution of the linear elliptic operator L in $\{0 \leq |\theta| < \theta_1\}$, $\theta_1 > \frac{\pi}{2}$,

(ii) v is $C^{0,\alpha}$ at 0 for some $0 < \alpha < 1$,

(iii) $v = 0$ on $\partial\{0 \leq |\theta| \leq \theta_1\}$.

The above conditions are modifications of the conditions for the barrier function of Laplace operator (see Lemma 4.3 of chapter 2 of [Fri]). A specific type of the barrier function satisfying the above conditions is already obtained by [Lee98] and used in [FS14]. However, in the following lemma, we are going to present a slightly different barrier function.

Lemma 3.3.6. *For fixed $\theta_1 > \frac{\pi}{2}$, $\alpha < 1$ and a linear operator $L = a_{ij}(x)D_{ij}$, ($\lambda I \leq A = (a_{ij}) \leq \Lambda I$), there are $\beta > 1$ and $\gamma > 0$ such that*

$$v(r, \theta) := r^{\alpha\beta}(e^{\gamma \cos \alpha\theta} - 1)$$

is a subsolution of L in $\{0 \leq |\theta| < \theta_1\}$.

Proof. Take $\beta > 1$ and $\gamma > 0$ such that $\alpha\beta = 1 - \epsilon, e^{\gamma} = \frac{\beta+1}{2}$, where $\epsilon > 0$ to be chosen later. By the definition of v and using the polar coordinate system, $x_1 = r\cos\theta, x_2 = r\sin\theta$, we know that

$$D^2 v = \begin{pmatrix} v_{11} & v_{12} & 0 & \dots & 0 \\ v_{21} & v_{22} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & 0 \end{pmatrix},$$

and

$$\begin{aligned}
v_{11} &= \cos^2\theta v_{rr} + \sin^2\theta \frac{1}{r^2} v_{\theta\theta} - \sin 2\theta \frac{1}{r} v_{r\theta} + \sin^2\theta \frac{1}{r} v_r + \sin 2\theta \frac{1}{r^2} v_\theta, \\
v_{22} &= \sin^2\theta v_{rr} + \cos^2\theta \frac{1}{r^2} v_{\theta\theta} + \sin 2\theta \frac{1}{r} v_{r\theta} + \cos^2\theta \frac{1}{r} v_r - \sin 2\theta \frac{1}{r^2} v_\theta, \\
v_{12} &= \sin\theta\cos\theta v_{rr} - \frac{\sin\theta\cos\theta}{r^2} v_{\theta\theta} + \cos 2\theta \frac{1}{r} v_{r\theta} - \frac{\cos\theta\sin\theta}{r} v_r - \frac{\cos 2\theta}{r^2} v_\theta,
\end{aligned}$$

where

$$\begin{aligned}
v_{rr} =&\alpha\beta(\alpha\beta-1)r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta}-1),\\
\frac{1}{r}v_r =&\alpha\beta r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta}-1),\\
\frac{1}{r^2}v_\theta =&-\alpha\gamma r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin\alpha\theta,\\
\frac{1}{r}v_{r\theta} =&-\alpha^2\beta\gamma r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin\alpha\theta,\\
\frac{1}{r^2}v_{\theta\theta} =&r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}(\alpha^2\gamma^2\sin^2\alpha\theta-\gamma\alpha^2\cos\alpha\theta).
\end{aligned}$$

Thus,

$$\begin{aligned}
v_{11} =& \cos^2\theta\alpha\beta(\alpha\beta-1)r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta}-1)+\sin^2\theta\alpha\beta r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta}-1)\\
&-\gamma\alpha^2\sin^2\theta\cos\alpha\theta r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}-\sin 2\theta(-\alpha^2\beta\gamma+\alpha\gamma)r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin\alpha\theta\\
&+\sin^2\theta\alpha^2\gamma^2r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin^2\alpha\theta
\end{aligned}$$

and

$$\begin{aligned}
v_{22} =& \sin^2\theta\alpha\beta(\alpha\beta-1)r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta}-1)+\cos^2\theta\alpha\beta r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta}-1)\\
&-\gamma\alpha^2\cos^2\theta\cos\alpha\theta r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}+\sin 2\theta(-\alpha^2\beta\gamma+\alpha\gamma)r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin\alpha\theta\\
&+\cos^2\theta\alpha^2\gamma^2r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin^2\alpha\theta.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Delta v =&\alpha\beta(\alpha\beta-1)r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta}-1)+\alpha\beta r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta}-1)-\gamma\alpha^2\cos\alpha\theta r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\\
&+\alpha^2\gamma^2r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin^2\alpha\theta\\
\geq&\alpha\beta(\alpha\beta-1)r^{\alpha\beta-2}(e^{\gamma}-1)\cos\alpha\theta+\alpha\beta r^{\alpha\beta-2}(\gamma\cos\alpha\theta)-\gamma\alpha^2\cos\alpha\theta r^{\alpha\beta-2}e^{\gamma}\\
&+\alpha^2\gamma^2r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin^2\alpha\theta\\
\geq&\{\alpha\beta(\alpha\beta-1)(e^\gamma-1)+\alpha\gamma(\beta-e^\gamma)\}\cos\alpha\theta r^{\alpha\beta-2}+\alpha^2\gamma^2r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin^2\alpha\theta\\
\geq&\left\{(1-\epsilon)(-\epsilon)+\alpha\log\left(\frac{\alpha+1-\epsilon}{2\alpha}\right)\right\}\left(\frac{-\alpha+1-\epsilon}{2\alpha}\right)\cos\alpha\theta r^{\alpha\beta-2}+\alpha^2\gamma^2r^{\alpha\beta-2}\sin^2\alpha\theta.
\end{aligned}$$

Let $C(\alpha,\epsilon):=\left\{(1-\epsilon)(-\epsilon)+\alpha\log\left(\frac{\alpha+1-\epsilon}{2\alpha}\right)\right\}\left(\frac{-\alpha+1-\epsilon}{2\alpha}\right)$. Then $C(\alpha,\epsilon)$ converges to $C(\alpha,0)>0$ as ϵ goes to 0. Thus, we know that there is a positive constant $\epsilon_0=\epsilon_0(\alpha)$ such that $C(\alpha,\epsilon)\geq C(\alpha,0)/2>0$, for all $\epsilon\leq\epsilon_0$. We may assume that $\epsilon\leq\epsilon_0$. Thus,

$$\Delta v\geq (C(\alpha,0)/2)\cos\alpha\theta r^{\alpha\beta-2}+\alpha^2\gamma^2r^{\alpha\beta-2}\sin^2\alpha\theta\geq 0.$$

Since $L = a_{ij}(x)D_{ij}$ and $\lambda I \le A = a_{ij}(x) \le \Lambda I$, we know that

$$Lv \ge \begin{cases} \lambda \Delta v \ge 0 & \text{if } v_{11}, v_{22} \ge 0, \\ \lambda v_{11} + \Lambda v_{22} & \text{if } v_{11} \ge 0 > v_{22}, \\ \Lambda v_{11} + \lambda v_{22} & \text{if } v_{22} \ge 0 > v_{11}. \end{cases}$$

Thus, to prove v is a subsolution of L in $\{0 \le |\theta| < \theta_1\}$, we are going to show $\lambda D_{11}v + \Lambda D_{22}v \ge 0$ and $\Lambda v_{11} + \lambda v_{22} \ge 0$ in $\{0 \le |\theta| < \theta_1\}$. Direct computations show that

$$\begin{aligned}
&\lambda D_{11}v + \Lambda D_{22}v = \lambda \Delta v \\
&\quad +(\Lambda - \lambda)\Big\{\sin^2\theta\alpha\beta(\alpha\beta - 1)r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta} - 1) + \cos^2\theta\alpha\beta r^{\alpha\beta-2}(e^{\gamma\cos\alpha\theta} - 1) \\
&\quad -\gamma\alpha^2\cos^2\theta\cos\alpha\theta r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta} + \sin 2\theta(-\alpha^2\beta\gamma + \alpha\gamma)r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin\alpha\theta \\
&\quad + \cos^2\theta\alpha^2\gamma^2 r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin^2\alpha\theta\Big\} \\
&\qquad \ge \lambda\Delta v \\
&\quad +(\Lambda - \lambda)\{\sin^2\theta(1-\epsilon)(-\epsilon)r^{\alpha\beta-2}(e^{\gamma} - 1)\cos\alpha\theta + \alpha\gamma(\beta - e^{\gamma})\cos^2\theta r^{\alpha\beta-2}(\cos\alpha\theta) \\
&\quad + \sin 2\theta\alpha\gamma\epsilon r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin\alpha\theta + \cos^2\theta\alpha^2\gamma^2 r^{\alpha\beta-2}e^{\gamma\cos\alpha\theta}\sin^2\alpha\theta\}. \\
&\quad \ge \Big[C(\alpha,\epsilon) + (\Lambda - \lambda)\Big\{\sin^2\theta(1-\epsilon)(-\epsilon)(e^{\gamma} - 1) + \alpha\gamma(\beta - e^{\gamma})\cos^2\theta\Big\}\Big] r^{\alpha\beta-2}\cos\alpha\theta \\
&\quad + \Big\{\alpha^2\gamma^2\sin^2\alpha\theta + (\Lambda - \lambda)\Big(\sin\alpha\theta\sin 2\theta\alpha\gamma\epsilon e^{\gamma\cos\alpha\theta} + \cos^2\theta\alpha^2\gamma^2\sin^2\alpha\theta\Big)\Big\} r^{\alpha\beta-2}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\Lambda v_{11} + \lambda v_{22} \\
&\quad \ge \Big[C(\alpha,\epsilon) + (\Lambda - \lambda)\Big\{\cos^2\theta(1-\epsilon)(-\epsilon)(e^{\gamma} - 1) + \alpha\gamma(\beta - e^{\gamma})\sin^2\theta\Big\}\Big] r^{\alpha\beta-2}\cos\alpha\theta \\
&\quad + \Big\{\alpha^2\gamma^2\sin^2\alpha\theta + (\Lambda - \lambda)\Big(-\sin\alpha\theta\sin 2\theta\alpha\gamma\epsilon e^{\gamma\cos\alpha\theta} + \sin^2\theta\alpha^2\gamma^2\sin^2\alpha\theta\Big)\Big\} r^{\alpha\beta-2}.
\end{aligned}$$

Since

$$\begin{aligned}
&(\Lambda - \lambda)\Big\{\sin^2\theta(1-\epsilon)(-\epsilon)(e^{\gamma} - 1) + \alpha\gamma(\beta - e^{\gamma})\cos^2\theta\Big\} \\
&\qquad = (\Lambda - \lambda)\left(\frac{1 - \alpha\epsilon}{2\alpha}\right)\Big\{\sin^2\theta(1-\epsilon)(-\epsilon) + \alpha\gamma\cos^2\theta\Big\}
\end{aligned}$$

and

$$\begin{aligned}
&(\Lambda - \lambda)\Big\{\cos^2\theta(1-\epsilon)(-\epsilon)(e^{\gamma} - 1) + \alpha\gamma(\beta - e^{\gamma})\sin^2\theta\Big\} \\
&= (\Lambda - \lambda)\left(\frac{1 - \alpha\epsilon}{2\alpha}\right)\Big\{\cos^2\theta(1-\epsilon)(-\epsilon) + \alpha\gamma\sin^2\theta\Big\}
\end{aligned}$$

converge to $\left(\frac{\Lambda-\lambda}{2}\right)\gamma\cos^2\theta$, $\left(\frac{\Lambda-\lambda}{2}\right)\gamma\sin^2\theta$, respectively, as ϵ goes to 0, we know that there exists $\tilde{\epsilon}=\epsilon(\alpha,\epsilon_0)$ such that

$$C(\alpha,\epsilon)+(\Lambda-\lambda)\left\{\sin^2\theta(1-\epsilon)(-\epsilon)(e^\gamma-1)+\alpha\gamma(\beta-e^\gamma)\cos^2\theta\right\}\geq C(\alpha,0)/4$$

and

$$C(\alpha,\epsilon)+(\Lambda-\lambda)\left\{\cos^2\theta(1-\epsilon)(-\epsilon)(e^\gamma-1)+\alpha\gamma(\beta-e^\gamma)\sin^2\theta\right\}\geq C(\alpha,0)/4,$$

for all $\epsilon\leq\tilde{\epsilon}$. We may assume that $\epsilon\leq\tilde{\epsilon}$ and $\epsilon\leq\epsilon_0$, i.e., we choose ϵ such that $\epsilon=\min\{\tilde{\epsilon},\epsilon_0\}$.

Take a sufficiently small constant $\epsilon'=\epsilon'(\alpha)>0$ such that

$$\frac{C(\alpha,0)}{4}\cos\alpha\epsilon'+(\Lambda-\lambda)\left(-4(\epsilon')^3\alpha\gamma e^\gamma\right)\geq 0.$$

1) Assume first that $-\epsilon'\leq\theta\leq\epsilon'$. Since $|\sin\alpha\theta|,|\sin 2\theta|\leq 2\epsilon'$, we know

$$\begin{aligned}
&\lambda D_{11}v+\Lambda D_{22}v\\
&\quad\geq\frac{C(\alpha,0)}{4}r^{\alpha\beta-2}\cos\alpha\theta\\
&\qquad+\left\{\alpha^2\gamma^2\sin^2\alpha\theta+(\Lambda-\lambda)\left(-4\epsilon'^3\alpha\gamma e^{\gamma\cos\alpha\theta}+\cos^2\theta\alpha^2\gamma^2\sin^2\alpha\theta\right)\right\}r^{\alpha\beta-2}\\
&\quad\geq\frac{C(\alpha,0)}{4}r^{\alpha\beta-2}\cos\alpha\epsilon'+\left\{(\Lambda-\lambda)\left(-4(\epsilon')^3\alpha\gamma e^\gamma\right)\right\}r^{\alpha\beta-2}\\
&\quad\geq\left\{\frac{C(\alpha,0)}{4}\cos\alpha\epsilon'+(\Lambda-\lambda)\left(-4(\epsilon')^3\alpha\gamma e^\gamma\right)\right\}r^{\alpha\beta-2}\\
&\quad\geq 0
\end{aligned}$$

and $\Lambda D_{11}v+\lambda D_{22}v\geq 0$.

2) Consider the other case, $-\theta_1\leq\theta\leq-\epsilon'$ or $\epsilon'\leq\theta\leq\theta_1$. Since $|\sin\alpha\theta|\geq\sin(\alpha\epsilon')$, we have

$$\begin{aligned}
&\alpha^2\gamma^2\sin^2\alpha\theta+(\Lambda-\lambda)\left(\pm\sin\alpha\theta\sin 2\theta\alpha\gamma\epsilon e^{\gamma\cos\alpha\theta}+\cos^2\theta\alpha^2\gamma^2\sin^2\alpha\theta\right)\\
&\geq\alpha^2\gamma^2\sin^2\alpha\epsilon'-(\Lambda-\lambda)\alpha\gamma e^\gamma\epsilon
\end{aligned}$$

and we may assume that ϵ is sufficiently small so that $\alpha^2\gamma^2\sin^2\alpha\epsilon'-(\Lambda-\lambda)\alpha\gamma e^\gamma\epsilon\geq 0$.

Therefore, in both cases, we know that v is a subsolution of L in $\{0<|\theta|<\theta_1\}$. $\square$

Proposition 3.3.7. *Let $u \in P_\infty(M)$ and assume that u is in $C^{2,\alpha}(\Omega(u))$ and F satisfies $(E1)$-$(E3)$ and $\Sigma = \{\mathcal{M} \,|\, F(\mathcal{M}) = 1\}$ satisfies $(E4)$ and $(E5)$. Suppose*

$$\delta_r(u) \geq \epsilon_0 > 0 \quad \textit{for all } r > 0.$$

Then u is a half-space solution, i.e., in an appropriate system of coordinates

$$u(x) = \frac{c'}{2}(x_1^+)^2,$$

where $1 = F(c' e_1 \otimes e_1)$ with $\frac{1}{\Lambda} \leq c' \leq \frac{1}{\lambda}$.

Proof. By the convexity of u, we know

$$\Lambda(u_\infty) = \{x \in \Lambda(u) : tx \in \Lambda(u) \quad \forall t > 0\},$$

where u_∞ is a shrink-down of u at 0, see Subsection 3.1.3 of this paper. By convexity of u and the fact that $\Lambda(u_\infty) \subset \Lambda(u)$, it is enough to show that $\Lambda(u_\infty)$ is a half plane.

Arguing by contradiction, suppose that $\Lambda(u_\infty)$ is not a half plane, i.e., we assume

$$\Lambda(u_\infty) \subset \{x \,|\, x = (\rho\cos\theta, \rho\sin\theta, x_3, \ldots, x_n), \theta_0 \leq |\theta| \leq \pi\} = \{\theta_0 \leq |\theta| \leq \pi\},$$

for some $\theta_0 > \pi/2$, in an appropriate system of coordinates. We choose θ_1 and α such that $\theta_0 > \theta_1 > \pi/2$, $\alpha\theta_1 = \pi/2$ and $0 < \alpha < 1$.

Let $w = D_1 u_\infty$. Since u is convex by Proposition 3.3.5, u_∞ is also convex and $w \geq 0$ in $\Omega(u_\infty)$. By differentiating $F(D^2 u_\infty(x)) = 1$ in $\Omega(u_\infty)$, w is a solution of $L = F_{ij}(D^2 u_\infty(x)) D_{ij}$ in $\Omega(u_\infty) \supset \{0 \leq |\theta| < \theta_1\}$ and $w \equiv 0$ on $\partial\Omega(u_\infty)$. Thus, by the maximum principle, $w > 0$ on $\{0 \leq |\theta| \leq \theta_1\} \cap \partial B_1 \Subset \Omega(u_\infty)$.

For the barrier function v with α, θ_1 and $a_{ij}(x) = F_{ij}(D^2 u_\infty(x))$ in Lemma 3.3.6, there exists a sufficiently small constant $c > 0$ such that

$$w \geq cv \text{ on } \{0 \leq |\theta| \leq \theta_1\} \cap \partial B_1.$$

Furthermore, we know that v is a subsolution of L in $\{0 < |\theta| < \theta_1\}$ and v vanishes on $\partial\{0 \leq |\theta| \leq \theta_1\}$. By the comparison principle, we have that $w \geq cv$ on $\{0 \leq |\theta| \leq \theta_1\} \cap B_1$. Thus, we arrive at a contradiction, since w is a Lipschitz function and v is $C^{\tilde{\alpha}}$ at 0 for some $0 < \tilde{\alpha} < 1$. □

Proposition 3.3.8. *Let $u \in P_1(M)$ with the obstacle function ϕ such that $-F(D^2\phi) \geq c > 0$ in B_1 and assume that F satisfies $(E1)$-$(E3)$ and Σ_t satisfies conditions in Theorem 3.2.2. Suppose*

$$\delta_r(u) \geq \epsilon_0 > 0, \quad \forall r < 1/4.$$

Then any blowup v_0 of $v = u - \phi$ at 0 is a half-space solution.

Proof. Let v_0 be a blowup of v at 0. By Lemma 3.3.3, we know that v_0 is in $P_\infty(M)$ for the fully nonlinear operator $G(\mathcal{M}) := \tilde{F}(\mathcal{M}, 0)/f(0)$ and $v_0 \in C^{2,\beta}(\{v_0 > 0\})$, for some $\beta \in (0,1)$. The conditions for F imply that G satisfies (E1)-(E3) and $\Sigma_G := \{\mathcal{M} \,|\, G(\mathcal{M}) = 1\} = \left\{\mathcal{M} \,|\, F(\mathcal{M} + D^2\phi(0)) = 0\right\} = \Sigma_F - D^2\phi(0)$ satisfies (E4) and (E5). Moreover, v_0 satisfies the thickness assumption,

$$\delta_r(v_0) \geq \epsilon_0 > 0, \quad \forall r > 0.$$

Then, by Proposition 3.3.7, v_0 is a half-space solution. □

3.3.3 Directional Monotonicity

The directional monotonicity for obstacle problems for linear operator and convex fully nonlinear operator is already discussed in [Lee98], [PSU], [FS14], [IM16a] and [LPS]. We give the proof for reader's convenience.

Lemma 3.3.9. *Let u be a solution of*

$$F(D^2u, rx) = f(rx)\chi_{\Omega(u)}, \quad u \geq 0 \quad \text{in } B_1, \tag{3.9}$$

with $\left\|D^2u\right\|_{L^\infty(B_1)} \leq M$, $f \in C^{0,1}(B_1)$, $f(x) \geq c > 0$ in B_1. Suppose that u is in $C^{2,\alpha}(\Omega(u))$ and we have

$$C\partial_e u - |\nabla u|^2 \geq -\epsilon_0 \quad \text{in } B_1,$$

for a direction e and a positive constant $\epsilon_0 < \frac{\lambda c^2}{2n\Lambda^3}$. Then there exists

$$r_0 = r_0(C, c, \|\nabla_x F\|_{L^\infty(B_M \times B_1)}, \|\nabla f\|_{L^\infty(B_1)}, M, \lambda, \Lambda)$$

such that

$$C\partial_e u - |\nabla u|^2 \geq 0 \quad \text{in } B_{1/2},$$

for all $0 < r \leq r_0'$.

proof. By differentiating (3.9), we have

$$F_{ij}(D^2u(y),ry)\partial_{ij}\nabla u = r\nabla f(ry) - r\nabla_x F(D^2u(y),ry), \qquad \text{a.e. in } \Omega(u).$$

Since $u \in C^{2,\alpha}(\Omega(u))$, the right side of the above equation is a L^∞ function and $\nabla u \in W^{2,p}_{loc}(\Omega(u))$ for any $p < \infty$.

Applying the linear operator $F_{ij}(D^2u(y),ry)\partial_{ij}$ to $|\nabla u|^2$ implies

$$\begin{aligned} &F_{ij}(D^2u(y),ry)\partial_{ij}|\nabla u|^2 \\ &\quad = 2\left(r\nabla f(ry) - r\nabla_x F(D^2u(y),ry)\right)\cdot\nabla u + 2F_{ij}(D^2u(y),ry)\partial_{ik}u\partial_{jk}u. \end{aligned}$$

By the uniform ellipticity for F_{ij} and F, we have

$$2F_{ij}(D^2u(y),ry)\partial_{ik}u\partial_{jk}u \geq 2\lambda|D^2u|^2$$

and

$$0 < c \leq f(rx) = F(D^2u,rx) = F(D^2u,rx) - F(0,rx) \leq \Lambda|D^2u|.$$

Hence we have

$$F_{ij}(D^2u(y),ry)\partial_{ij}|\nabla u|^2 \geq 2\left(r\nabla f(ry) - r\nabla_x F(D^2u(y),ry)\right)\cdot\nabla u + \frac{2\lambda c^2}{\Lambda^2}.$$

Arguing by contradiction, suppose there is a point $y \in B_{1/2}\cap\Omega(u)$ such that $C\partial_e u(y) - |\nabla u(y)|^2 < 0$. Define the auxiliary function

$$\phi(x) = C\partial_e u(x) - |\nabla u(x)|^2 + \frac{4\lambda c^2}{\Lambda^2}\frac{1}{2\Lambda n}|x-y|^2.$$

Then, by the above inequalities, there exists

$$r_0 = r_0(C, c, \|\nabla_x F\|_{L^\infty(B_M\times B_1)}, \|\nabla f\|_{L^\infty(B_1)}, M, \lambda, \Lambda),$$

such that

$$F_{ij}(D^2u(x),rx)\partial_{ij}\phi \leq 0,$$

on $B_{1/2}(y)\cap\Omega(u)$, for all $r \leq r_0$.

Since $\phi(y) < 0$, by the maximum principle, ϕ has the negative infimum on $\partial(B_{1/2}(y)\cap\Omega(u))$ and moreover we have $\phi \geq 0$ on $\partial\Omega(u)\cap B_{1/2}(y)$. Thus we obtain

$$\inf_{\partial B_{1/2}(y)\cap\Omega(u)} \phi < 0$$

and it is equivalent to

$$\inf_{\partial B_{1/2}(y)\cap\Omega(u)} \left(C\partial_e u - |\nabla u|^2\right) < -\frac{4\lambda c^2}{\Lambda^2}\frac{1}{2\Lambda n}\frac{1}{4} = -\frac{\lambda c^2}{2n\Lambda^3}.$$

Since $\epsilon_0 < \frac{\lambda c^2}{2n\Lambda^3}$, we have a contradiction. □

Lemma 3.3.10. *Let $u \in C^{2,\alpha}(\Omega(u))$ be a solution of*

$$F(D^2u, x) = f(x)\chi_{\Omega(u)}, \quad u \geq 0 \quad \textit{in } B_1,$$

with $0 \in \Gamma(u)$ and $\left\|D^2u\right\|_{L^\infty(B_1)} \leq M$, $f \in C^{0,1}(B_1)$, $f(x) \geq c > 0$ in B_1. Let

$$u_0(x) = \frac{1}{2}(x_1^+)^2,$$

where u_0 is a blowup of u at 0. Then for any $\delta \in (0, 1]$ there exists

$$r_\delta = r_\delta(u, \delta, c, \|\nabla_x F\|_{L^\infty(B_M\times B_1)}, \|\nabla f\|_{L^\infty(B_1)}, E, \lambda, \Lambda)$$

such that

$$\partial_e u \geq 0 \quad \textit{in } B_{r_\delta} \quad \textit{for any } e \in C_\delta \cap \partial B_1,$$

where

$$C_\delta = \{x \in \mathbb{R}^n : x_1 > \delta|x'|\}, \quad x' = (x_2, ..., x_n).$$

Proof. Since $\|u_r - u_0\|_{C^1(B_1)} \to 0$ as $r \to 0$ and u_0 satisfies

$$2\delta^{-1}\partial_e u_0 - |\nabla u_0|^2 \geq 0 \text{ in } B_1 \quad \text{for any } e \in C_\delta \cap \partial B_1,$$

there exists $r'_\delta = r'_\delta(\delta, u)$ such that for $r < r'_\delta$,

$$2\delta^{-1}\partial_e u_r - |\nabla u_r|^2 \geq -\frac{\lambda c^2}{2n\Lambda^3} \quad \text{in } B_1.$$

By Lemma 3.3.9, there exists

$$r_\delta = r_\delta(u, \delta, c, \|\nabla_x F\|_{L^\infty(B_M\times B_1)}, \|\nabla f\|_{L^\infty(B_1)}, M, \lambda, \Lambda),$$

such that for $r \leq r_\delta$,

$$\partial_e u_r \geq (\delta/2)|\nabla u_r|^2 \geq 0 \quad \text{in } B_{1/2} \quad \text{for any } e \in C_\delta \cap \partial B_1$$

and

$$\partial_e u \geq 0 \quad \text{in } B_{r_\delta} \quad \text{for any } e \in C_\delta \cap \partial B_1.$$

□

By the directional monotonicity for u (Lemma 3.3.10), we have the uniqueness of blowup (see Proposition 4.6 of [PSU]).

Proposition 3.3.11 (Uniqueness of blowup)**.** *Let u be as in Lemma 3.3.10. Then the blowup function of u at 0 is unique, i.e., in an appropriate system of coordinates, for any sequence $r_i \to 0$ as $i \to \infty$,*

$$u_{r_i} \to u_0 = \frac{1}{2}(x_1^+)^2 \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^n).$$

3.3.4 Proof of Theorem 3.1.1 and Corollary 3.1.2

The methods for the rest part of the proof of Theorem 3.1.1 and Corollary 3.1.2 are straightforward, since it is already discussed in [Fri], [Lee98], [PSU], [FS14], [IM16a] and [LPS].

Lemma 3.3.12. *Let u, F and Σ_t be as in Theorem 3.1.1 and let r_1 be as in Lemma 3.3.10. Then for $x^0 \in \Gamma(u) \cap B_{r_1} = \Gamma(v) \cap B_{r_1}$, the blowup function of $v = u - \phi$ at x^0 is a half-space function.*

Proof. By Theorem 3.2.2, $u \in C^{2,\alpha}(\Omega(u))$. Thus, we know that $v \in C^{2,\alpha}(\Omega(v))$ and by Proposition 3.3.8, v_0 is a half-space solution. Then by Lemma 3.3.10, we have the directional monotonicity for v, i.e., for any $\delta \in (0, 1]$, there exists

$$r_\delta = r_\delta(v, \delta, c, \|DF\|_{L^\infty(B_{M+\|F(D^2\phi)\|_{L^\infty(B_1)}})}, \|\nabla f\|_{L^\infty(B_1)}, M, \lambda, \Lambda),$$

such that

$$\partial_e v \geq 0 \quad \text{in } B_{r_\delta} \quad \text{for any } e \in C_\delta.$$

Then the free boundary $\Gamma(v) \cap B_{r_1}$ is represented by Lipschitz functions.

For $x^0 \in \Gamma(v) \cap B_{r_1}$, the Lipschitz regularity of $\Gamma(v)$ implies the thickness condition for v at x_0, i.e., for some $\epsilon_0, \tilde{r} = \tilde{r}(x^0) > 0$,

$$\delta_r(v, x_0) \geq \epsilon_0 > 0, \quad \forall \tilde{r} \geq r > 0.$$

Then, by Proposition 3.3.8, we know that the blowup function of v at x is a half-space solution. □

Proof of Theorem 3.1.1 and Corollary 3.1.2. By the directional monotonicity for v (Lemma 3.3.10), we know that the free boundary $\Gamma(v) \cap B_{r_\delta}$ is represented as a graph $x_n = f(x')$ with Lipschitz constant of f not exceeding δ. Since $\delta > 0$ is arbitrary, we have the tangent plane of $\Gamma(u)$ with the normal vector e_n at 0. By Lemma 3.3.12, every point z in $\Gamma(v) \cap B_{r_1}$ has the tangent plane. Let $\nu(z)$ is the interior normal vector of $\Omega(u)$ at z. Then by Lemma 3.3.10, $e \cdot \nu(z) \geq 0$ for any $z \in B_{r_\delta} \cap \Gamma(v)$ and $e \in C_\delta \cap \partial B_1$. Hence, $\nu(z) \in C_{1/\delta} \cap \partial B_1$ and

$$|e_1 - \nu(z)| \leq \sqrt{2\left(1 - \sqrt{\frac{1}{1+\delta^2}}\right)} \quad , \text{for all } z \in B_{r_\delta} \cap \Gamma(v).$$

Hence, we know that $\Gamma(v)$ is C^1 at 0 and by the same method, $\Gamma(v)$ is C^1 in B_{r_1}.

By the same argument as in the proof of Theorem 3.10 of Section 2 of [Fri], we have $u \in C^2(\overline{\Omega(u)} \cap B_\rho)$, for sufficiently small $\rho > 0$. Furthermore, $v = u - \phi$ is a solution of

$$\hat{F}(D^2 v, x) = F(D^2 v + D^2 \phi) = 0 \quad \text{ in } \Omega(u) = \Omega(v),$$

where $\hat{F}(\mathcal{M}, x) = F(\mathcal{M} + D^2\phi(x))$. $\hat{F}(0,0) = F(D^2\phi(0)) < 0$ and $v \equiv 0$, $Dv \equiv 0$ on $\Gamma(v)$. Therefore, by using Hodograph-Legendre transformation, we have the statement of the corollary, see e.g. Theorem 1' of [KN] and Section 1 of Chapter 2 of [Fri]. $\square$

Chapter 4

Double Obstacle Problem (Linear Case)

4.1 Introduction

In the study of the double obstacle problem for Laplacian in a domain $D \subset \mathbb{R}^n$, we consider the following partial differential equations:

$$\begin{cases} \Delta u \geq 0, & \text{in } \{u > \phi_1\} \cap D, \\ \Delta u \leq 0, & \text{in } \{u < \phi_2\} \cap D, \\ \phi_1(x) \leq u(x) \leq \phi_2(x) & \text{in } D, \\ u(x) = g(x) & \text{on } \partial D, \end{cases} \tag{4.1}$$

with $\phi_1, \phi_2 \in C^{1,1}(\overline{D})$, $\partial D \in C^{2,\alpha}$, $g \in C^{2,\alpha}(\overline{D})$ and $\phi_1 \leq g \leq \phi_2$ in ∂D. In Chapter 5, the existence, uniqueness and the optimal regularity of the solution of the double obstacle problem for fully nonlinear operator is proved. Since fully nonlinear operator is containing the Laplacian, we omit the proof of the results for the Laplace case. In the study of the regularity of the free boundary of the double obstacle problem for Laplacian, the ACF monotonicity formula and Weiss' monotonicity formula is used and they are not applied for the fully nonlinear case. So, we will present another method to have the regularity of the free boundary for fully nonlinear case in Chapter 5.

In order to study the regularity of the free boundaries of the double obstacle problem, we first present the reduced problem of (4.1) :

$$\Delta u = f\chi_{\{0<u<\psi\}} + \Delta\psi\chi_{\{0<u=\psi\}}, \quad 0 \leq u \leq \psi \quad \text{in } B_1, \tag{4.2}$$

where $f \in C^{0,1}(B_1)$, with *the upper obstacle function* $\psi \in C^{1,1}(B_1) \cap C^{2,1}(\overline{\Omega(\psi)})$, $\Omega(\psi) := B_1 \setminus (\{\psi = 0\} \cap \{\nabla\psi = 0\})$, see Subsection 5.1.1. In this chapter, we prove the regularity of the free boundary for more general problem than (4.2). That is the reduced problem, (4.2), with out the sign assumption for u, $u \geq 0$ which is called *the nosign double obstacle problem*:

$$\Delta u = f\chi_{\Omega(u)\cap\{u<\psi\}} + \Delta\psi\chi_{\Omega(u)\cap\{u=\psi\}}, \qquad u \leq \psi \quad \text{in } B_1, \tag{4.3}$$

where $f \in C^{0,1}(B_1)$, $\Omega(u) = B_1 \setminus (\{u = 0\} \cap \{\nabla u = 0\})$ with *the upper obstacle* ψ such that

$$\psi \in C^{1,1}(B_1) \cap C^{2,1}(\overline{\Omega(\psi)}), \quad \Omega(\psi) = B_1 \setminus (\{\psi = 0\} \cap \{\nabla\psi = 0\}) .$$

Thus the regularity of the free boundaries of (4.2) is obtain as a corollary.

Now, we explain the method to have the regularity of the free boundaries. In Section 4.3, we prove that if u is the global solution of (4.3) with the upper obstacle $\psi = c(x_1^+)^2$, then u is a two-dimensional function, by using the ACF monotonicity formula. Furthermore we prove that if the global solution u with the upper obstacle $\psi = c(x_1^+)^2$ is homogeneous degree two then u is also a half space solution, $c(x_1^+)^2$. In Section 4.4, we have the directional monotonicity: if u is a solution of (4.3) and the blowup u_0 and ψ_0 of u and ψ at a free boundary point 0 are the half space functions, $c(x_1^+)^2$, then u and $\partial_e u$ are positive near the origin, where e is close to e_1. In Section 4.5, we prove that if u is a solution of (4.3) with thickness assumption for u and ψ, then the blowup of u and the upper obstacle at a free boundary point 0 are the half space functions, $c(x_1^+)^2$, by using the results in Section 4.3. Finally in Section 4.6, the directional monotonicity which we obtained in Section (4.4) implies the regularity of the free boundary.

4.1.1 Notation

We will use the following notations throughout the paper.

C, C_0, C_1	generic constants
χ_E	the characteristic function of the set E, $(E \subset \mathbb{R}^n)$
$\overline{E}$	the closure of E
∂E	the boundary of a set E
$\lvert E\rvert$	$n-$ dimensional Lebesgue measure of the set E
$B_r(x), B_r$	$\{y \in \mathbb{R}^n : \lvert y-x\rvert < r\}, \quad B_r(0)$
$\Omega(u), \Omega(\psi)$	see Equation (4.3)
$\Lambda(u), \Lambda(\psi)$	$B_1 \setminus \Omega(u), B_1 \setminus \Omega(\psi)$
$\Omega^\psi(u)$	$B_1 \setminus (\{u=\psi\} \cap \{\nabla u = \nabla\psi\}) = B_1 \setminus \{u=\psi\} = \{u > \psi\}$
	$(u \le \psi$ implies $\{u=\psi\} \cap \{\nabla u = \nabla\psi\} = \{u=\psi\}.)$
$\Lambda^\psi(u)$	$B_1 \setminus \Omega^\psi(u) = \{u = \psi\}$
$\Gamma(u), \Gamma^\psi(u)$	$\partial\Lambda(u) \cap B_1, \partial\Lambda^\psi(u) \cap B_1$
$\Gamma^d(u)$	$\Gamma(u) \cap \Gamma^\psi(u)$
u^+, u^-	$\max(u,0), \max(-u,0)$
$\lVert u\rVert_{\infty,E}$	the supremum norm of the function u on the set E
$\partial_\nu, \partial_{\nu e}$	first and second directional derivatives
$P_r(M), P_\infty(M)$	see Definition 4.1.2, 4.1.3
$\delta_r(u,x), \delta_r(u)$	see Definition 4.1.1

4.1.2 Preliminaries

Let u be a solution of (4.3) in B_r. Then a *rescaling function* of u at x_0 with $\lambda > 0$ is

$$u_\lambda(x) = u_{\lambda,x_0}(x) := \frac{u(x_0+\lambda x) - u(x_0)}{\lambda^2}, \quad x \in B_{r/\lambda}.$$

The $C^{1,1}$-regularity of solution u (Theorem 4.2.1) implies the uniform boundedness of $C^{1,1}$-norm of the rescaling functions and the uniform boundedness gives limit functions which are called a blowup and a shrink-down. More precisely, if u is a solution of (4.3) in B_r, then for a sequence $\lambda_i \to 0$, there exists a subsequence λ_{i_j} of λ_i and $u_0 \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that

$$u_{\lambda_{i_j}} \to u_0 \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^n) \quad \text{for any } 0 < \alpha < 1.$$

Such u_0 is called a *blowup of u at x_0*. Let u be a solution of (4.3) in $\mathbb{R}^n$. Then, for a sequence $\lambda_i \to \infty$, there exists a subsequence λ_{i_j} of λ_i and $u_0 \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that

$$u_{\lambda_{i_j}} \to u_\infty \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^n) \quad \text{for any } 0 < \alpha < 1.$$

Such u_∞ is called a *shrink-down of u at x_0*.

Definition 4.1.1. We denote by $\delta_r(u, x)$ the thickness of $\Lambda(u)$ on $B_r(x)$, i.e.,

$$\delta_r(u, x) := \frac{\mathrm{MD}(\Lambda(u) \cap B_r(x))}{r},$$

where $\mathrm{MD}(A)$ is the least distance between two parallel hyperplanes containing A. We will use the abbreviated notation $\delta_r(u)$ for $\delta_r(u, 0)$.

Remark. The thickness δ_r satisfies $\delta_1(u_r) = \delta_r(u)$, where $u_r = u_{r,0}$. Thus, by the fact that $\limsup_{r\to 0} \Lambda(u_r) \subset \Lambda(u_0)$, we have

$$\limsup_{r\to 0} \delta_r(u) \le \delta_1(u_0).$$

Hence the thickness assumption (4.4) in Theorem 4.1.1 implies

$$\min\{\delta_r(u_0), \delta_r(\psi_0)\} \ge \epsilon_0 \quad \forall r > 0,$$

for any blowups u_0 and ψ_0 of u and ψ at 0, respectively.

In order to state our main results, we define classes of local and global solutions of the problem.

Definition 4.1.2. (Local solutions) We say a function u belongs to the class $P_r(M)$ $(0 < r < \infty)$, if u satisfies :

(i) $\Delta u = f\chi_{\Omega(u)\cap\{u<\psi\}} + \Delta\psi\chi_{\Omega(u)\cap\{u=\psi\}}, \quad u \le \psi \quad \text{in } B_r,$

(ii) $\|D^2 u\|_{\infty, B_r} \le M,$

(iii) $0 \in \Gamma^d(u),$

where $f \in C^{0,1}(B_r)$ and $\psi \in C^{1,1}(B_r) \cap C^{2,1}(\overline{\Omega(\psi)})$.

Definition 4.1.3. (Global solutions) We say a function u belongs to the class $P_\infty(M)$, if u satisfies with a constant $a > 1$:

(i) $\Delta u = \chi_{\Omega(u)\cap\{u<\psi\}} + a\chi_{\Omega(u)\cap\{u=\psi\}}, \quad u \le \psi \quad \text{in } \mathbb{R}^n$,

(ii) $\Delta\psi = a\chi_{\Omega(\psi)}$ in $\mathbb{R}^n$,

(iii) $\|D^2 u\|_{\infty,\mathbb{R}^n} \le M$,

(iv) $0 \in \Gamma(u)$.

4.1.3 Main Results

Theorem 4.1.1. *(Regularity of free boundaries) Let $u \in P_1(M)$ with an upper obstacle ψ such that*

$$0 \in \partial\Omega(\psi), \qquad \lim_{x\to 0, x\in\Omega(\psi)} \Delta\psi(x) > f(0), \quad f \ge c > 0 \textit{ in } B_1,$$

and

$$\inf\{\Delta\psi, \Delta\psi - f\} \ge c > 0 \textit{ in } \Omega(\psi).$$

Suppose

$$\min\{\delta_r(u), \delta_r(\psi)\} \ge \epsilon_0 \quad \forall r < 1/4. \tag{4.4}$$

Then there is $r_0 = r_0(u,\psi) > 0$ such that $\Gamma(u) \cap B_{r_0}$ and $\Gamma^{\psi}(u) \cap B_{r_0}$ are C^1 graphs.

4.2 Standard Results

4.2.1 Optimal regularity

Since the optimal regularity of the solution of u of the obstacle problem for double obstacle problem which containing the Laplacian case, (4.1) is discussed in Chapter 5, we omit the proof of the regularity. Instead of that, in this subsection, we discuss the optimal regularity of the solution of (4.3).

The double obstacle problem (4.3) is in a more general class of problems, studied in [FS14, IM16a], where optimal regularity of solutions for the class is already studied. Thus we state the result with a simple application.

Theorem 4.2.1. *(Optimal regularity) Let u be a $W^{2,n}$ solution of* (4.3) *in B_1, with $f \in C^{0,\alpha}(B_1)$ and $\psi \in C^{1,1}(B_1)$. Then*

$$\left\|D^2 u\right\|_{\infty,B_{1/2}} \le C$$

where $C > 0$ is a universal constant.

Proof. Since $\psi \in C^{1,1}(B_1)$, we obtain that $|D^2u|$ is bounded a.e. on $\{u = \psi\}$. Then the solution u of (4.3) satisfies

$$\begin{cases} \Delta u = f & \text{a.e. in } B_1 \cap (\Omega(u) \cap \{u < \psi\})\,, \\ |D^2u| \le K & \text{a.e. in } B_1 \setminus (\Omega(u) \cap \{u < \psi\})\,, \end{cases}$$

for a positive constant K; i.e., u is in the general classes defined in [FS14, IM16a]. By the $C^{1,1}$ regularity theory in the papers (more specifically, Theorem 1.2 of [FS14], Theorem 2.1 of [IM16a]), we obtain the $C^{1,1}$ regularity of the solution u. □

4.2.2 Non-degeneracy

Non-degeneracy is one of the important properties of the obstacle problem. In particular, it implies that the blowups of the solutions are still solutions to the problem and that the blowups of the solutions are not the identically zero function. Moreover, it implies that the Lebesgue measure of the free boundary is zero.

Lemma 4.2.2. *Let $u \in P_1(M)$. If $f \ge c > 0$ in B_1 and $\Delta\psi \ge c > 0$ in $\Omega(\psi)$, then*

$$\sup_{\partial B_r(x)} u \ge u(x) + \frac{c}{8n} r^2, \quad x \in \overline{\Omega(u)} \cap B_1,$$

for any $B_r(x) \Subset B_1$.

Proof. (i) Let $x^0 \in \Omega(u) \cap B_1$ be such that $u(x^0) > 0$. Consider the auxiliary function

$$\phi(x) := u(x) - u(x^0) - \frac{c}{2n}|x - x^0|^2.$$

Due to $\Omega(u) \cap \{u = \psi\} \subset \Omega(\psi)$ and the assumptions for f and $\Delta\psi$, we obtain

$$\Delta u = f\chi_{\Omega(u)\cap\{u<\psi\}} + \Delta\psi\chi_{\Omega(u)\cap\{u=\psi\}} \ge c \quad \text{in } \Omega(u). \tag{4.5}$$

Hence we have

$$\Delta\phi \ge \Delta u - c \ge 0 \text{ on } B_r(x^0) \cap \Omega(u).$$

Thus, by the maximum principle, ϕ attains its maximum on $\partial(B_r(x^0) \cap \Omega(u))$. Hence

$$0 = \phi(x^0) \le \sup_{\partial(B_r(x^0)\cap\Omega(u))} \phi.$$

Moreover, $\phi(x) = -u(x^0) - \frac{c}{2n}|x - x^0|^2 < 0$ on $\partial\Omega(u)$, which implies that

$$0 \le \sup_{\partial B_r(x^0)\cap\Omega(u)} \phi,$$

and

$$\sup_{\partial B_r(x^0)} u \ge u(x^0) + \frac{c}{2n}r^2.$$

(ii) Now, let $x^0 \in \Omega(u) \cap B_1$ and assume $u(x^0) \le 0$. Suppose that there is a point $x^1 \in B_{r/2}(x^0)$ such that $u(x^1) > 0$. Then we obtain

$$\sup_{B_r(x^0)} u \ge \sup_{B_{r/2}(x^1)} u \ge u(x^1) + \frac{c}{8n}r^2 \ge u(x^0) + \frac{c}{8n}r^2.$$

Since u is subharmonic,

$$\sup_{\partial B_r(x^0)} u = \sup_{B_r(x^0)} u \ge u(x^0) + \frac{c}{8n}r^2.$$

Suppose that $u(x) \le 0$ in $B_{r/2}(x^0)$. By the maximum principle, we know that $u(x) \equiv 0$ in $B_{r/2}(x^0)$ or $u(x) < 0$ in $B_{r/2}(x^0)$. The first case is impossible, since $x^0 \in \Omega(u)$. The second case implies that $\Delta u \ge c$ in $B_{r/2}(x^0)$. By using the auxiliary function $w(x) = u(x) - \frac{c|x-x_0|^2}{2n}$, we obtain

$$\sup_{\partial B_{r/2}(x^0)} w \ge \sup_{B_{r/2}(x^0)} w \ge w(x^0) = u(x^0),$$

and thus

$$\sup_{\partial B_{r/2}(x^0)} u \ge u(x^0) + \frac{c}{8n}r^2.$$

Since u is subharmonic, we have the desired inequality.

Let $x^0 \in \partial\Omega(u) \cap B_1$ and take a sequence of points $x^j \in \Omega(u)$ such that $x^j \to x^0$ as $j \to \infty$. By passing to the limit as j goes to ∞, we have the desired inequality for $x^0 \in \overline{\Omega(u)} \cap B_1$. □

By using the non-degeneracy for u, we have the local porosity for $\partial\Lambda(u) = \Gamma(u)$*. Moreover, the porosity implies* $\Gamma(u)$ *has a Lebesgue measure zero (see Section 3.2.1 of [PSU]).*

Lemma 4.2.3. *[Lebesgue measure of* $\Gamma(u)$*] Let* $u \in P_1(M)$*. If* $f \ge c > 0$ *in* B_1 *and* $\Delta\psi \ge c > 0$ *on* $\Omega(\psi)$*, then* $\Gamma(u)$ *has a Lebesgue measure zero.*

Remark. By the non-degeneracy, we know that $0 \in \Gamma(u_0)$ where u_0 is a blowup of $u \in P_1(M)$ (see Theorem 3.17 (iv) of [PSU]). However, we do not have any information whether $0 \in \Gamma^{\psi_0}(u_0)$, where ψ_0 is a blowup of the upper obstacle ψ of u (which is the reason why we assume (iv) in Definition 4.1.3 and not $0 \in \Gamma^d(u) = \Gamma(u) \cap \Gamma^\psi(u)$).

However, we have $0 \in \Gamma^{\psi_0}(u_0)$, under the additional assumption for $u \in P_1(M)$, $0 \leq u$ in B_1 and $\Delta\psi - f \geq c > 0$ and $\Delta\psi \geq c > 0$ in $\Omega(\psi)$. If we assume $0 \leq u$ in B_1, then u is a solution of

$$\Delta u = f\chi_{\{0<u<\psi\}} + \Delta\psi\chi_{\{0<u=\psi\}}, \qquad 0 \leq u \leq \psi \quad \text{in } B_1,$$

and $v := \psi - u$ is a solution of

$$\Delta v = (\Delta\psi - f)\chi_{\{0<v<\psi\}} + \Delta\psi\chi_{\{0<v=\psi\}}, \qquad 0 \leq v \leq \psi \quad \text{in } B_1.$$

Since $\Delta\psi - f$ lies in $C^{0,1}(\overline{\Omega(\psi)}) = C^{0,1}(\overline{\{\psi > 0\}})$ but not in $C^{0,1}(B_1)$, we know that v does not belong to $P_1(M)$. However, $0 \leq v \leq \psi$ implies $\{v > 0\} \subset \{\psi > 0\}$ and

$$\Delta v = (\Delta\psi - f)\chi_{\{0<v<\psi\}} + \Delta\psi\chi_{\{0<v=\psi\}} \geq c \quad \text{in } \Omega(v) = \{v > 0\},$$

provided $\Delta\psi - f \geq c$ and $\Delta\psi \geq c$ in $\Omega(\psi) = \{\psi > 0\}$. Then the rest of the proof for the non-degeneracy for v is a repetition of the arguments in the proof of Lemma 4.2.2. Thus, we have the non-degeneracy for v and moreover $0 \in \Gamma(v_0) = \Gamma^{\psi_0}(u_0)$ and $|\Gamma(v)| = |\Gamma^\psi(u)| = 0$.

4.3 Properties of Global Solutions

In this section, we consider some properties of global solutions with the upper obstacle $\psi = \frac{a}{2}(x_1^+)^2$.

4.3.1 Dimensionality Reduction and Positivity of Global Solutions with the Upper Obstacle $\psi = \frac{a}{2}(x_1^+)^2$

In order to discuss dimensionality reduction of global solutions, we introduce Alt-Caffarelli-Friedman (ACF) monotonicity formula which is an important tool in analysis of regularity of free boundary; see [ACF], and also [CS] for a more detailed proof.

Theorem 4.3.1 (Alt-Caffarelli-Friedman (ACF) monotonicity formula). *Let $u_\pm$ be continuous functions on B_1 such that*

$$u_\pm \geq 0, \quad \Delta u_\pm \geq 0, \quad u_+ \cdot u_- = 0 \quad \text{in } B_1$$

Then the functional

$$r \to \Phi(r) = \Phi(r, u_+, u_-) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_+|^2}{|x|^{n-2}} dx \int_{B_r} \frac{|\nabla u_-|^2}{|x|^{n-2}} dx$$

is nondecreasing for $0 < r < 1$.

Theorem 4.3.2 (Equality in ACF monotonicity formula). *Let $u_\pm$ be as in Theorem 4.3.1 and assume that $\Phi(r_1) = \Phi(r_2)$ for some $0 < r_1 < r_2 < 1$. Then either one of the following holds:*

(i) $u_+ = 0$ *in* B_{r_2} *or* $u_- = 0$ *in* B_{r_2}*;*

(ii) there exists a unit vector e and constants $k_\pm > 0$ such that

$$u_+(x) = k_+(x \cdot e)^+, \quad u_-(x) = k_-(x \cdot e)^- \quad \text{in } B_{r_2}.$$

Lemma 4.3.3. *Let $u \in P_1(M)$ with the upper obstacle $\psi = \frac{a}{2}(x_1^+)^2$. Then for any unit vector e such that $e \perp e_1$,*

$$\Delta(\partial_e u)^\pm \geq 0 \quad \text{in } B_1.$$

Proof. Let e be a unit vector such that $e \perp e_1$ and $E := \{\partial_e u > 0\}$. Since $\partial_e \psi \equiv 0$, we know that $E \subset \Omega(u) \cap \{u < \psi\}$ ($u \leq \psi$ implies $\{u = \psi\} = \{\{u = \psi\} \cap \{\nabla u = \nabla \psi\}\}$) and $\Delta u = 1$ on E. Consequently, we have $\Delta(\partial_e u) = 0$ on E and

$$\Delta(\partial_e u)^+ \geq 0 \quad \text{in } B_1$$

This is left to the reader as an exercise.

We have the same inequality for $(\partial_e u)^-$, by using the direction $-e$ instead of e. □

Lemma 4.3.4. *Let $u \in P_\infty(M)$ with the upper obstacle*

$$\psi(x) = \frac{a}{2}(x_1^+)^2 \quad \text{in } \mathbb{R}^n.$$

Assume that there exists $\epsilon_0 > 0$ such that

$$\delta_r(u) \geq \epsilon_0 \quad \forall r > 0.$$

Then we have $|Int\Lambda(u)| \neq 0$ and u is two-dimensional, i.e.

$$u(x) = w(x_1, x_2) \quad \forall x \in \mathbb{R}^n,$$

with $\partial_2 w \geq 0$, in an appropriate system of coordinates.

Proof. Suppose $|Int\Lambda(u)| = 0$. Then, by Lemma 4.2.3, we have $|\partial\Lambda(u)| = 0$ and $|\Lambda(u)| = 0$. Thus u is a solution of

$$\Delta u = \chi_{\{u<\psi\}} + a\chi_{\{u=\psi\}} \quad \text{a.e. in } \mathbb{R}^n.$$

Define $\tilde{\psi} := \frac{a}{2}(x_1)^2$. Then $\tilde{v} := \tilde{\psi} - u$ is a solution of

$$\Delta \tilde{v} = (a-1)\chi_{\{u<\psi\}} \quad \text{a.e. in } \mathbb{R}^n.$$

Since $\Omega(u) \cap \{u = \psi\} \subset \{x_1 \geq 0\}$, we know that $\Delta u \leq 1$ a.e. in $\{x_1 < 0\}$. On the other hand, $\Delta u = a$ a.e. in $\{x_1 < 0\} \cap \{u = \tilde{\psi}\} \cap \{\nabla u = \nabla\tilde{\psi}\}$. Therefore, we know that $|\{x_1 < 0\} \cap \{u = \tilde{\psi}\} \cap \{\nabla u = \nabla\tilde{\psi}\}| = 0$. By the definition of $\tilde{\psi}$ and ψ, we obtain $\{u = \tilde{\psi}\} \cap \{\nabla u = \nabla\tilde{\psi}\} = \{u = \psi\} \cap \{\nabla u = \nabla\psi\} = \{u = \psi\}$ a.e. in $\mathbb{R}^n$ ($u \leq \psi$ implies the last equality). Therefore $\tilde{v}$ is a solution of

$$\Delta \tilde{v} = (a-1)\chi_{\Omega(\tilde{v})} \quad \text{a.e. in } \mathbb{R}^n,$$

where $\Omega(\tilde{v}) := \mathbb{R}^n \setminus (\{\tilde{v} = 0\} \cap \{\nabla\tilde{v} = 0\}) = \mathbb{R}^n \setminus \left(\{u = \tilde{\psi}\} \cap \{\nabla u = \nabla\tilde{\psi}\}\right)$.

By the definition of $\tilde{v}$, we know that $0 \in \Lambda(\tilde{v})$. Suppose $0 \in int\Lambda(\tilde{v})$. Then there is a ball B_r such that $\tilde{v} \equiv 0$ and $u \equiv \tilde{\psi}$ in $\mathbb{R}^n$. Thus we have a contradiction to $\delta_r(u) > \epsilon_0$ for all $r > 0$.

Suppose $0 \in \Gamma(\tilde{v})$. Let $u_0, \tilde{v}_0$ be blowup functions of u and $\tilde{v}$, respectively, such that $u_0 = \tilde{\psi} - \tilde{v}_0$. Then $\tilde{v}_0$ is a solution of

$$\Delta \tilde{v}_0 = (a-1)\chi_{\Omega(\tilde{v}_0)} \quad \text{a.e. in } \mathbb{R}^n,$$

and by Theorem 3.22 of [PSU], we know that $\tilde{v}_0$ is a polynomial or a half-space solution. In the both cases, we have a contradiction to $\delta_r(u_0) > \epsilon_0$ for all $r > 0$. Thus, we obtain

$$|Int\Lambda(u)| \neq 0.$$

Let u_∞ be a shrink-down of u at 0, then $u_\infty \in P_\infty(M)$ with the upper obstacle $\psi = \frac{a}{2}(x_1^+)^2$ and the thickness assumption,

$$\min\{\delta_r(u_\infty), \delta_r(\psi)\} > \epsilon_0 \quad \forall r > 0.$$

Hence we also have

$$|Int\Lambda(u_\infty)| \neq 0.$$

For $r > 0$ and a unit vector e, we define

$$\phi_e(r, u) := \Phi(r, (\partial_e u)^+, (\partial_e u)^-).$$

By $W^{2,p}$ convergence $u_{r_j} \to u_\infty$, we have

$$\phi_e(r, u_\infty) = \lim_{j\to\infty} \phi_e(r, u_{r_j}).$$

Additionally, we obtain the rescaling property,

$$\phi_e(r, u_{r_j}) = \phi_e(rr_j, u).$$

By Lemma 4.3.3, we know that $(\partial_e u)^\pm$ and $(\partial_e u_\infty)^\pm$ satisfy the assumptions in ACF monotonicity formula (Theorem 4.3.1), for any unit vector e such that $e \perp e_1$. Thus we know that the limit $\phi_e(\infty, u)$ exists and

$$\phi_e(r, u_\infty) = \lim_{j\to\infty} \phi_e(rr_j, u) = \phi_e(\infty, u),$$

for all $r > 0$ and $e \perp e_1$, i.e., $\phi_e(r, u_\infty)$ is constant for all $r > 0$ and $e \perp e_1$. By Theorem 4.3.2, either one of the following holds for $e \perp e_1$:

(i) $(\partial_e u_\infty)^+ \equiv 0$ or $(\partial_e u_\infty)^- \equiv 0$ in $\mathbb{R}^n$;

(ii) there exists a unit vector $w = w(e)$ and constants $k_\pm = k_\pm(e) > 0$ such that

$$(\partial_e u_\infty)^+ = k_+(x\cdot w)^+, \quad (\partial_e u_\infty)^- = k_-(x\cdot w)^- \quad \forall x \in \mathbb{R}^n.$$

Since $|Int\Lambda(u_\infty)| \neq 0$, we know that (ii) does not hold for any direction $e \perp e_1$, i.e., we know that (i) holds for any direction $e \perp e_1$. Consequently, we have that

$$0 \le \phi_e(r, u) \le \phi_e(\infty, u) = \phi_e(r, u_\infty) = 0,$$

for any $r > 0$ and $e \perp e_1$. Then again, by $|Int\Lambda(u)| \neq 0$ and Theorem 4.3.2, we know that $\partial_e u$ has a sign for all $e \perp e_1$, i.e.,

$$\partial_e u \ge 0 \quad \text{or} \quad \partial_e u \le 0 \quad \text{in } \mathbb{R}^n \text{ for any } e \perp e_1.$$

By Lemma 4.3.5, in an appropriate system of coordinates

$$u(x) = w(x_1, x_2), \quad x \in \mathbb{R}^n,$$

with $\partial_2 w \geq 0$. $\square$

Lemma 4.3.5. *If $u \in C^1(\mathbb{R}^n)$ and if $\partial_e u$ does not change sign in $\mathbb{R}^n$, where $e \perp e_1$, then there exist a function $w \in C^1(\mathbb{R}^2)$ and a direction $\tilde{e} \perp e_1$ such that*

$$u(x) = w(x_1, x \cdot \tilde{e}), \quad x \in \mathbb{R}^n$$

where w is a monotone function with the second variable.

Proof. The obvious proof is left to the reader. $\square$

Proposition 4.3.6. *Let $u \in P_\infty(M)$ with the upper obstacle*

$$\psi(x) = \frac{a}{2}(x_1^+)^2 \quad \textit{in } \mathbb{R}^n.$$

Assume that there exists $\epsilon_0 > 0$ such that

$$\delta_r(u) \geq \epsilon_0 \quad \forall r > 0.$$

Then $0 \leq u$ in $\mathbb{R}^2$ and u is a solution of

$$\Delta u = \chi_{\{0<u<\psi\}} + a\chi_{\{0<u=\psi\}}, \quad 0 \leq u \leq \psi \quad \textit{a.e. in } \mathbb{R}^2. \tag{4.6}$$

Proof. By Lemma 4.3.4, we know u is a 2-dimensional function and $|Int\Lambda(u)| \neq 0$ and in an appropriate system of coordinates

$$\partial_2 u(x) \geq 0 \quad \forall x \in \mathbb{R}^2.$$

Thus we know that there is a ball $B_\delta(x_0) \subset \Lambda(u)$ and $u \leq 0$ in

$$K(x^0, \delta) = \{(x_1, x_2 - m) | (x_1, x_2) \in B_\delta(x^0), m \geq 0\}.$$

Since u is subharmonic, by the strong maximum principle, we obtain

$$u \equiv 0 \quad \text{in } K(x^0, \delta).$$

By the assumption, $\partial_2 u \geq 0$ in $\mathbb{R}^2$, we know that the limit, $\lim_{x_2 \to -\infty} u(x_1, x_2)$ exists, for all $x_1 \in \mathbb{R}^1$. Then we define a 1-dimensional function

$$\hat{u}(x_1) := \lim_{x_2 \to -\infty} u(x_1, x_2).$$

Since $K(x^0, \delta) \subset \Lambda(u)$, we obtain

$$|u(x_1, x_2)| \leq \frac{M}{2}|x_1 - x_1^0|^2,$$

where $x_2 \leq x_2^0$, and therefore

$$|\hat{u}(x_1)| \leq \frac{M}{2}|x_1 - x_1^0|^2,$$

and $\hat{u}(x_1)$ is finite for any $x_1 \in \mathbb{R}^1$.

By the definition of $\hat{u}$ and the fact that $u(x_1, x_2 - t)$ is a solution of (4.6) for all $t > 0$, we know that $\hat{u}$ is a limit of the solutions of (4.6) and $\hat{u}$ is a solution of the obstacle problem with upper obstacle $\psi(x_1) = \frac{a}{2}(x_1^+)^2$ in $\mathbb{R}^1$.

By the definition of $\hat{u}$ and $K(x^0, \delta) \subset \Lambda(u)$, we know $B'_\delta(x_1^0) \subset \Lambda(\hat{u})$. Suppose that the connected component of $\Lambda(\hat{u})$ containing $B'_\delta(x_1^0)$ is a closed interval, $[\alpha, \beta] \subset \mathbb{R}^1$ (call it $\tilde{\Lambda}(\hat{u})$). By the non-degeneracy, we know that there are points α_0 and β_0 such that $\alpha_0 < \alpha < \beta < \beta_0$ and $\hat{u}(x) > 0$ for all $x \in (\alpha_0, \alpha) \cup (\beta, \beta_0)$. Thus, if there is a point z such that $\hat{u}(z) < 0$, then there is an open interval I such that $\hat{u} > 0$ on I and $\hat{u} = 0$ at the ends points of I. By the maximum principle, however, $\hat{u} \leq 0$ on I. Thus, we arrive at a contradiction. In the case that $\tilde{\Lambda}(\hat{u})$ is $(-\infty, \alpha]$ or $[\beta, \infty)$ for some $\alpha, \beta \in \mathbb{R}^1$, we also have the same contradiction. Therefore we obtain $\hat{u} \geq 0$ in $\mathbb{R}^1$.

By the definition of $\hat{u}$ and $\partial_2 u \geq 0$ in $\mathbb{R}^n$, we obtain

$$u(x_1, x_2) \geq \hat{u}(x_1) \geq 0 \quad \forall x = (x_1, x_2) \in \mathbb{R}^2,$$

and u is a solution of

$$\Delta u = \chi_{\{0<u<\psi\}} + a\chi_{\{0<u=\psi\}}, \quad 0 \leq u \leq \psi \quad \text{a.e. in } \mathbb{R}^2.$$

$\square$

4.3.2 Homogeneity of Blowup and Shrink-down of Global Solutions with the Upper Obstacle $\psi = \frac{a}{2}(x_1^+)^2$

In order to deal with homogeneity, we introduce Weiss' energy functional for the problem (4.3). It is a modification of Weiss' energy functional for the classical obstacle problem, $\Delta u = \chi_{\{u>0\}}$, $u \geq 0$ in B_R, and has already appeared in [Ale]. We give the proof for reader's convenience.

Definition 4.3.1. Let $u \in P_R(M)$ be a solution of

$$\Delta u = \chi_{\{0<u<\psi\}} + \Delta\psi\chi_{\{0<u=\psi\}} \quad \text{on } B_R,$$

with the upper obstacle

$$\psi(x) = \frac{a}{2}(x_1^+)^2.$$

We define Weiss' energy functional for u and $0 < r < R$ as

$$\begin{aligned} W(r,u) : &= \frac{1}{r^{n+2}} \int_{B_r} (|Du|^2 + 2u\Delta u)dx - \frac{2}{r^{n+3}} \int_{\partial B_r} u^2 dH^{n-1} \\ &= \frac{1}{r^{n+2}} \int_{B_r} |Du|^2 dx + \int_{B_r \cap \{\psi > u > 0\}} 2udx + \int_{B_r \cap \{\psi = u > 0\}} 2audx \\ &- \frac{2}{r^{n+3}} \int_{\partial B_r} u^2 dH^{n-1} \end{aligned}$$

Theorem 4.3.7 (Weiss' monotonicity formula)**.** *Let u, ψ be as in Definition 4.3.1. Then $r \to W(r,u)$ is a nondecreasing absolutely continuous function for $0 < r < R$ and*

$$\frac{d}{dr}W(r,u) = \frac{2}{r^{n+4}} \int_{\partial B_r} |x \cdot Du(x) - 2u(x)|^2 dH^{n-1},$$

for a.e. $0 < r < R$. Furthermore, if $W(r,u)$ is constant for $r > 0$, then u is homogeneous of degree two, i.e.,

$$u(\lambda x) = \lambda^2 u(x) \quad \textit{for all } x \in \mathbb{R}^n, \lambda > 0.$$

Proof. By the scaling property $W(r,u) = W(1,u_r)$, we have

$$\begin{aligned} \frac{d}{dr}W(r,u) &= \frac{d}{dr}W(1,u_r) \\ &= \int_{B_1} \frac{d}{dr}(|Du_r|^2)dx + \int_{B_1 \cap \{\psi_r > u_r > 0\}} 2\frac{d}{dr}(u_r)dx \\ &+ \int_{B_1 \cap \{\psi_r = u_r > 0\}} 2a\frac{d}{dr}(u_r)dx - 2\int_{\partial B_1} \frac{d}{dr}(u_r^2)dH^{n-1}. \end{aligned}$$

Since $\frac{d}{dr}(\nabla u_r) = \nabla\frac{du_r}{dr}$ and $\frac{du_r}{dr} = \frac{x \cdot \nabla u_r - 2u_r}{r}$, we obtain, by integration by parts,

$$
\begin{aligned}
\frac{d}{dr}W(r,u) &= 2\int_{B_1} -\Delta u_r \frac{du_r}{dr}dx + \int_{B_1\cap\{\psi_r>u_r>0\}} 2\frac{du_r}{dr}dx \\
&\quad + \int_{B_1\cap\{\psi_r=u_r>0\}} 2a\frac{du_r}{dr}dx + 2\int_{\partial B_1}(\partial_\nu u_r - 2u_r)\frac{du_r}{dr}dH^{n-1} \\
&= 2r\int_{\partial B_1}\left|\frac{du_r}{dr}\right|^2 dH^{n-1}.
\end{aligned}
$$

Then we have the desired equality after scaling. □

Corollary 4.3.8. *(Homogeneity of blowup and shrink-down) Let $u \in P_\infty(M)$ be a solution of*

$$\Delta u = \chi_{\{0<u<\psi\}} + \Delta\psi\chi_{\{0<u=\psi\}} \quad \text{on } \mathbb{R}^n,$$

with the upper obstacle

$$\psi(x) = \frac{a}{2}(x_1^+)^2.$$

Then any blowup function u_0 of u at 0 and any shrink-down u_∞ of u at 0 are homogeneous of degree two.

Proof. Suppose that $\lambda_j \to 0$ as $j \to \infty$ and $u_{\lambda_j} \to u_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$ as $j \to \infty$. Then for $r > 0$

$$W(r,u_0) = \lim_{j\to\infty} W(r,u_{\lambda_j}) = \lim_{j\to\infty} W(\lambda_j r,u) = W(0+,u),$$

i.e., $W(r,u_0)$ is constant for any r. Hence, u_0 is homogeneous of degree two.

In order to prove the homogeneity for shrink-down u_∞, we take a sequence $\lambda'_j \to \infty$ as $j \to \infty$ and $u_{\lambda'_j} \to u_\infty$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$ as $j \to \infty$. The same argument as above shows that $W(r,u_\infty)$ is constant for any $r > 0$ and the homogeneity of shrink-down. □

Under the conditions of Theorem 4.1.1, we know that the blowups and shrink-downs of the blowups u_0 of $u \in P_1(M)$ are two-dimensional and homogeneous of degree two, see the proof of Proposition 4.5.1. For further study on the main theorem, we need to know about the global solutions which are two-dimensional and homogeneous of degree two.

Lemma 4.3.9. *Let $u \in P_\infty(M)$ and u is a solution of*

$$\Delta u = \chi_{\{0<u<\psi\}} + a\chi_{\{0<u=\psi\}}, \quad 0 \le u \le \psi \quad \textit{a.e. in } \mathbb{R}^2,$$

with the upper obstacle

$$\psi(x) = \frac{a}{2}(x_1^+)^2,$$

for a constant $a > 1$. Suppose that u is homogeneous of degree two. Then

$$u(x) = \frac{1}{2}(x_1^+)^2 \quad \textit{or} \quad u(x) = \frac{a}{2}(x_1^+)^2.$$

Proof. By the condition $0 \le u \le \psi$ and $\psi(x) = \frac{a}{2}(x_1^+)^2$, we know that $\{x_1 < 0\} \subset \{u = 0\}$. We claim that $\partial_2 u \equiv 0$ in $\mathbb{R}^2$, i.e., u is one-dimensional function.

Assume that $\{\partial_2 u \neq 0\} \cap \{x_1 > 0\} \neq \emptyset$. Then by the homogeneity of degree one for $\partial_2 u$, we know that there is a cone

$$C := \{r\theta \mid r > 0, \alpha_1 < \theta < \alpha_2\} \subset \{x_1 > 0\},$$

($-\frac{\pi}{2} \le \alpha_1 < \alpha_2 \le \frac{\pi}{2}$) such that $\partial_2 u \neq 0$ in C and $\partial_2 u = 0$ on ∂C. Since $\partial_2 \psi \equiv 0$, we know that

$$C \subset \{0 < u < \psi\} \cap \{x_1 > 0\}$$

and $\partial_2 u$ is harmonic on C. Hence $\partial_2 u := rf(\theta)$ satisfies

$$\Delta \partial_2 u = \Delta\left(rf(\theta)\right) = \frac{1}{r}\left(f(\theta) + f''(\theta)\right) = 0 \quad \text{on } C.$$

Thus $f(\theta)$ satisfies $-f''(\theta) = f(\theta)$ in (α_1, α_2) and $f(\theta) = 0$ on $\partial(\alpha_1, \alpha_2)$. Hence we obtain $f(\theta) = c\cos(\theta)$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$, $C = \{r\theta \mid r > 0, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\} = \{x_1 > 0\}$ and

$$\partial_2 u = cr\cos(\theta) = cx_1 \text{ in } \{x_1 > 0\}.$$

Then $u = cx_1x_2$ in $\{x_1 > 0\} = \{0 < u < \psi\}$. It is a contradiction to $\Delta u = 1$ in $\{0 < u < \psi\}$. Hence we obtain that $\partial_2 u \equiv 0$ in $\mathbb{R}^2$. This completes the proof. □

4.4 Directional Monotonicity

In this section, we prove the directional monotonicity for solutions to (4.3). Basically, the proof for the directional monotonicity for the double obstacle problem is similar to that of classical obstacle problem. However, we need to check the details of the proof, since in the double obstacle case, we need to consider the upper obstacle.

Lemma 4.4.1. *Let $u \in P_1(M)$ and $f \geq c > 0$ in B_1, $\Delta\psi \geq c > 0$ in $\Omega(\psi)$ and any blowup u_0 satisfies*

$$u_0(x) = \frac{1}{2}(x_1^+)^2 \quad or \quad u_0 = \frac{a}{2}(x_1^+)^2.$$

Suppose, further, that

$$\|u - u_0\|_{L^\infty(B_1)} \leq \epsilon.$$

Then

$$u > 0 \quad in\ \{x_1 > \sqrt{2\epsilon}\} \cap B_1,$$
$$u = 0 \quad in\ \left\{x_1 \leq -4\sqrt{\frac{n\epsilon}{c}}\right\} \cap B_{1/2}.$$

The proof is exactly the same as that of Lemma 4.1 of [PSU]. Hence, we omit the proof.

Lemma 4.4.2. *Let $u \in P_1(M)$ and $f \geq c > 0$ in B_1, $\Delta\psi \geq c > 0$ in $\Omega(\psi)$. Suppose that we have*

$$C\partial_e\psi - \psi \geq -\epsilon_0, \quad C\partial_e u - u \geq -\epsilon_0 \quad in\ B_1,$$

for a direction e and $\epsilon_0 < c/64n$. Then we obtain

$$C\partial_e\psi - \psi \geq 0 \quad in\ B_{3/4}, \quad C\partial_e u - u \geq 0 \quad in\ B_{1/2},$$

where $\|Df\|_{L^\infty(\overline{B_1})}, \left\|D^3\psi\right\|_{L^\infty(\overline{\Omega(\psi)\cap B_1})} < \dfrac{c}{2C}$.

Proof. First, we will prove

$$C\partial_e\psi - \psi \geq 0 \quad \text{in } B_{3/4}.$$

Arguing by contradiction, suppose there is a point $y \in B_{3/4} \cap \Omega(\psi)$ such that $C\partial_e\psi(y) - \psi(y) < 0$. Define the auxiliary function

$$\phi(x) = C\partial_e\psi(x) - \psi(x) + \frac{c}{4n}|x - y|^2.$$

Then

$$\begin{aligned}
\Delta\phi(x) &= C\Delta\partial_e\psi(x) - \Delta\psi(x) + \frac{c}{2} \\
&\leq C\left\|D^3\psi\right\|_{L^\infty(\overline{\Omega(\psi)\cap B_1})} - \Delta\psi(x) + \frac{c}{2} \\
&\leq c - \Delta\psi \leq 0
\end{aligned}$$

on $B_{1/4}(y)\cap\Omega(\psi)$. Since $\phi(y)<0$, by the minimum principle, ϕ has the negative infimum on $\partial(B_{1/4}(y)\cap\Omega(\psi))$. Since $\phi\geq 0$ on $\partial\Omega(\psi)$, we have

$$\inf_{\partial B_{1/4}(y)\cap\Omega(\psi)}\phi<0.$$

It is equivalent to

$$\inf_{\partial B_{1/4}(y)\cap\Omega(\psi)}(C\partial_e\psi-\psi)<-\frac{c}{64n}.$$

Since $\epsilon_0<c/64n$, we have a contradiction.

By using $C\partial_e\psi-\psi\geq 0$ in $B_{3/4}$, $\{u=\psi\}=\{u=\psi\}\cap\{\nabla u=\nabla\psi\}$ (since $u\leq\psi$) and the same method as above, we have

$$C\partial_e u-u\geq 0\quad\text{ in }B_{1/2}\cap\Omega(u)\cap\{u<\psi\}.$$

This completes the proof of the lemma. $\square$

Lemma 4.4.3. *Let $u\in P_1(M)$ and $f\geq c>0$ in B_1, $\Delta\psi\geq c>0$ in $\Omega(\psi)$. Let further*

$$\psi_0=\frac{a}{2}(x_1^+)^2$$

and

$$u_0(x)=\frac{1}{2}(x_1^+)^2\quad or\quad u_0=\frac{a}{2}(x_1^+)^2.$$

Suppose also $\|Df\|_{L^\infty(\overline{\{\psi>0\}\cap B_1})},\left\|D^3\psi\right\|_{L^\infty(\overline{\{\psi>0\}\cap B_1})}<\dfrac{c\delta}{2}$ for $0<\delta\leq 1$ and

$$\|u-u_0\|_{C^1(B_1)},\quad\|\psi-\psi_0\|_{C^1(B_1)}\leq\epsilon.\tag{4.7}$$

Then $\epsilon\leq\dfrac{c}{128n}$ implies

$$u\geq 0\quad in\ B_{1/2},$$

and $\epsilon\leq\dfrac{c\delta}{128n}$ implies

$$\partial_e u\geq 0\quad in\ B_{1/2},$$

for any

$$e\in C_\delta\cap\partial B_1,$$

where

$$C_\delta=\{x\in\mathbb{R}^n:x_1>\delta|x'|\},\quad x'=(x_2,...,x_n).$$

Proof. Direct computation shows that

$$\delta^{-1}\partial_e u_0 - u_0 \geq 0, \quad \delta^{-1}\partial_e \psi_0 - \psi_0 \geq 0 \text{ in } B_1 \quad \text{for any } e \in C_\delta \cap \partial B_1.$$

By using the closeness condition (4.7) for $\epsilon \leq c\delta/128n$, we have

$$\delta^{-1}\partial_e u - u \geq -2\epsilon\delta^{-1} \geq -\frac{c}{64n}, \quad \delta^{-1}\partial_e \psi - \psi \geq -\frac{c}{64n} \quad \text{in } B_1.$$

By Lemma 4.4.2, we have

$$\delta^{-1}\partial_e u - u \geq 0 \quad \text{in } B_{1/2} \quad \text{for any } e \in C_\delta \cap \partial B_1. \tag{4.8}$$

Recalling Lemma 4.4.1, we have

$$u = 0 \quad \text{in } \left\{x_1 \leq -\frac{1}{2\sqrt{2}}\right\} \cap B_{1/2}.$$

Let $\delta = 1$ and multiply (4.8) by $exp(-e \cdot x)$. Then we have

$$\partial_e(exp(-e \cdot x) \cdot u) \geq 0 \quad \text{in } B_{1/2}.$$

By integrating $(exp(-e \cdot x) \cdot u)$ with direction $e \in C_1$, we obtain $u \geq 0$ in $B_{1/2}$. Moreover, we have that $\partial_e u \geq 0$ in $B_{1/2}$, for any $e \in C_\delta \cap \partial B_1$. □

The rescaled function u_r at 0 satisfies

$$\Delta u_r = f(rx)\chi_{\{\psi_r > u_r > 0\}} + \Delta\psi(rx)\chi_{\{\psi_r = u_r > 0\}} \quad \text{in } B_{1/r}.$$

Moreover, when r tends to 0, then u_r converges to u_0 in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$ and

$$\|D(f(rx))\|_{L^\infty(B_1)} = r\,\|Df(rx)\|_{L^\infty(B_1)} \leq r\,\|Df(x)\|_{L^\infty(B_1)},$$

$$\|D(\Delta\psi(rx))\|_{L^\infty(B_1)} = r\,\|D\Delta\psi(rx)\|_{L^\infty(B_1)} \leq r\,\|D\Delta\psi(x)\|_{L^\infty(B_1)}$$

converge to 0. Therefore, we have the following lemma.

Lemma 4.4.4. *(Directional monotonicity) Let $u \in P_1(M)$ and $f \geq c > 0$ in B_1, $\Delta\psi \geq c > 0$ in $\Omega(\psi)$. Let*

$$\psi_0 = \frac{a}{2}(x_1^+)^2$$

and

$$u_0(x) = \frac{1}{2}(x_1^+)^2 \quad or \quad u_0 = \frac{a}{2}(x_1^+)^2,$$

where u_0 and ψ_0 are blowup functions of u and ψ, respectively. Then for any $\delta \in (0, 1]$ there exists $r_\delta = r(\delta, u) > 0$ such that

$$u \geq 0 \quad in\ B_{r_1}$$

$$\partial_e u \geq 0 \quad in\ B_{r_\delta} \quad for\ any\ e \in C_\delta.$$

4.5 Classification of Blowups

In this section, we classify the blowups by using the results in Section 4.3, 4.4.

Proposition 4.5.1. *Let $u \in P_1(M)$ with an upper obstacle ψ such that*

$$0 \in \partial\Omega(\psi), \qquad \lim_{x\to 0, x\in\Omega(\psi)} \Delta\psi(x) = a > f(0) = 1, \quad f \geq c > 0 \textit{ in } B_1,$$

and

$$\inf\{\Delta\psi, \Delta\psi - f\} \geq c > 0 \textit{ in } \Omega(\psi).$$

Suppose

$$\min\{\delta_r(u), \delta_r(\psi)\} \geq \epsilon_0 \quad \forall r < 1/4.$$

Then

$$\psi_0 = \frac{a}{2}(x_1^+)^2 \quad \textit{and} \quad u_0 = \frac{1}{2}(x_1^+)^2 \quad \textit{in } \mathbb{R}^n,$$

in an appropriate system of coordinates.

Proof. Let u_0, ψ_0 be a global solution of u, ψ, respectively. Then ψ_0 is a global solution of

$$\Delta\psi_0 = a\chi_{\Omega(\psi_0)} \quad \text{in } \mathbb{R}^n,$$

with the thickness assumption,

$$\delta_r(\psi_0) > \epsilon_0, \quad \forall r > 0.$$

By the non-degeneracy for ψ (the proof is almost the same as that of Lemma 4.2.2), we know $0 \in \Gamma(\psi_0)$; see also Proposition 3.17 (iv) in [PSU]. By Theorem II of [CKS], we obtain that ψ_0 is a half-space solution, i.e.,

$$\psi_0 = \frac{a}{2}(x_1^+)^2 \quad \text{in } \mathbb{R}^n,$$

in an appropriate system of coordinates. By Proposition 4.3.6, u_0 is two-dimensional, $u_0(x) = u_0(x_1, x_2)$, and hence a solution of

$$\Delta u_0 = \chi_{\{0<u_0<\psi_0\}} + a\chi_{\{0<u_0=\psi_0\}}, \quad 0 \leq u_0 \leq \psi_0 \quad \text{a.e. in } \mathbb{R}^2.$$

Let $u_{00} = (u_0)_0$ and $u_{0\infty} = (u_0)_\infty$ be blowup and respectively shrink-down of u_0 at 0. By Corollary 4.3.8, $u_{00}, u_{0\infty}$ are homogeneous of degree two and by Lemma 4.3.9,

$$u_{00} = \frac{1}{2}(x_1^+)^2 \quad \text{or} \quad \frac{a}{2}(x_1^+)^2 \quad \text{and} \quad u_{0\infty} = \frac{1}{2}(x_1^+)^2 \quad \text{or} \quad \frac{a}{2}(x_1^+)^2.$$

By Lemma 4.4.1, 4.4.2 and 4.4.3 for u_{00} and u_0 and the fact that $(u_0)_r$ converges to u_{00} as $r \to 0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n)$, we know there are $r', \epsilon' > 0$ such that

$$\delta^{-1}\partial_e u_0 - u_0 \geq 0 \quad \text{in } B_{r'} \quad \text{for any } e \in C_\delta \cap \partial B_1. \tag{4.9}$$

$$u_0 = 0 \quad \text{in } \{x_1 < -\epsilon'\} \cap B_{r'}. \tag{4.10}$$

Moreover, by Lemma 4.4.1, 4.4.2 and 4.4.3 for u_0 and u in $B_{r'}$ with the conditions, (4.9) and (4.10), we know that there is r'' such that

$$u \geq 0 \quad \text{in } B_{r''}.$$

Then we know that $0 \in \Gamma^{\psi_0}(u_0)$ and $0 \in \Gamma^{\psi_0}(u_{00}), \Gamma^{\psi_0}(u_{0\infty})$ (see Remark 4.2.2). Thus we obtain

$$u_{00} = u_{0\infty} = \frac{1}{2}(x_1^+)^2.$$

Since

$$\begin{aligned} W(1, u_{00}) &= \lim_{r\to 0} W(1, (u_0)_r) = \lim_{r\to 0} W(r, u_0) \\ &\leq \lim_{r\to\infty} W(r, u_0) = \lim_{r\to\infty} W(1, (u_0)_r) = W(1, u_{0\infty}) \end{aligned}$$

and $W(1, u_{00}) = W(1, u_{0\infty})$, we know that $W(r, u_0)$ is constant for $r > 0$. Hence, by Lemma 4.3.9 and $0 \in \Gamma^{\psi}(u_0)$, we know that u_0 is homogeneous of degree two and

$$u_0(x) = \frac{1}{2}(x_1^+)^2.$$

□

4.6 Proof of Theorem 4.1.1

Let u be as in Proposition 4.5.1. Then a blowup function u_0 of u at 0 is a half-space solution, i.e.,

$$u_0 = \frac{1}{2}(x_1^+)^2,$$

in an appropriate system of coordinates. By the directional monotonicity for u (Lemma 4.4.4), we have the uniqueness of blowup (see Proposition 4.6 of [PSU]).

Proposition 4.6.1 (Uniqueness of blowup). *Let u be as in Proposition 4.5.1. Then the blowups of u at 0 is unique, i.e., in an appropriate system of coordinates, for any sequence $\lambda_i \to 0$,*

$$u_{\lambda_i} \to u_0 = \frac{1}{2}(x_1^+)^2 \quad \text{in } C^{1,\alpha}_{loc}(\mathbb{R}^n)$$

as $\lambda_i \to 0$.

Lemma 4.6.2. *Let u be as in Proposition 4.5.1. Then there is $r_1' = r_1'(u,\psi) > 0$ such that the blowup function of u at $x \in \Gamma(u) \cap B_{r_1'}$ are half-space functions.*

Proof. By Proposition 4.5.1, we have the directional monotonicity for u (see Lemma 4.4.4). Moreover, by Lemma 4.4.2 we also have the directional monotonicity for ψ. Thus, for any $\delta \in (0,1]$, there exists $r_1 \geq r_\delta' = r_\delta'(u,\psi) > 0$ such that

$$\begin{aligned} \psi, u \geq 0 \quad &\text{in } B_{r_1'} \\ \partial_e \psi, \partial_e u \geq 0 \quad &\text{in } B_{r_\delta'} \quad \text{for any } e \in C_\delta. \end{aligned}$$

Hence, by the sign condition $u \geq 0$ in $B_{r_1'}$, we know that u is a solution of

$$\Delta u = f\chi_{\{0<u<\psi\}} + \Delta\psi\chi_{\{0<u=\psi\}}, \quad 0 \leq u \leq \psi \quad \text{in } B_{r_1'}$$

and the free boundaries $\partial\{u = 0\} \cap B_{r_1'} = \Gamma(u) \cap B_{r_1'}$ and $\partial\{\psi = 0\} \cap B_{r_1'}$ are represented by Lipschitz functions; for details, see Proposition 4.8 of [PSU].

Case 1) Let $x^0 \in \Gamma(u) \cap B_{r_1'} = \partial\{u = 0\} \cap B_{r_1'}$ and assume that there exists $r_0 > 0$ such that

$$\{u = \psi\} \cap B_r(x^0) \neq \emptyset \quad \forall r < r_0.$$

Then we can find a sequence of points $x^j \in \{u = \psi\}$ converging to x^0 as $j \to \infty$. Then we have

$$\psi(x^j) = u(x^j) \to 0 \quad \text{as } j \to \infty,$$

i.e., $x^0 \in \{\psi = 0\}$. By the sign condition $0 \leq u \leq \psi$ in $B_{r'}$, we know $\{\psi = 0\} \subset \{u = 0\}$ in $B_{r_1'}$ and therefore $x^0 \in \partial\{u = 0\} \cap B_{r_1'}$ implies $x^0 \in \partial\{\psi = 0\}$. On the other hand, Lipschitz regularity of $\partial\{u = 0\}$ and $\partial\{\psi = 0\}$ implies the thickness condition for ψ and u, i.e., for some $\epsilon_0, \tilde{r} = \tilde{r}(x^0) > 0$,

$$\min\{\delta_r(u), \delta_r(\psi)\} \geq \epsilon_0 > 0 \quad \forall \tilde{r} \geq r > 0.$$

Then, by Proposition 4.5.1, we know that the blowup function of u at x^0 is a half-space solution (we may assume $\lim_{x\to x^0, x\in\Omega(\psi)} \Delta\psi(x) > f(x^0)$, by the conditions $\psi \in C^{1,1}(B_1) \cap C^{2,1}(\overline{\Omega(\psi)})$, $f \in C^{0,1}(B_1)$ and $\lim_{x\to 0, x\in\Omega(\psi)} \Delta\psi(x) = a > f(0) = 1$).

Case 2) Let $x^0 \in \Gamma(u)$ and assume that there exists $r_0 > 0$ such that

$$\{u = \psi\} \cap B_{r_0}(x^0) = \emptyset.$$

Then u is a solution of an obstacle problem

$$\Delta u = f\chi_{\{u>0\}}, \quad u \geq 0 \quad \text{in } B_{r_0}(x^0).$$

By Theorem II of [CKS] and the thickness condition for u at x^0, we know that that the blowup function of u at 0 is a half-space solution. $\square$

Proof of Theorem 4.1.1. By Proposition 4.5.1, we have the directional monotonicity for u (see Lemma 4.4.4). Thus, we know that the free boundary $\Gamma(u) \cap B_{r\delta/2}$ is represented as a graph $x_n = f(x')$ with Lipschitz constant of f not exceeding δ. Since $\delta > 0$ is arbitrary, we have a tangent plane of $\Gamma(u)$ and the normal vector e_n at 0. By Lemma 4.6.2, we know that every point $z \in \Gamma(u) \cap B_{r_1'}$ has a tangent plane. Moreover again, by using the directional monotonicity, we obtain that $\Gamma(u) \cap B_{r_1'}$ is C^1 (see Theorem 4.10 of [PSU]).

We know that there is a ball $B_{r_1'}$ such that $u \geq 0$ in $B_{r_1'}$ and $v = \psi - u$ is a solution of

$$\Delta v = (\Delta\psi - f)\chi_{\{0<v<\psi\}} + \Delta\psi\chi_{\{0<v=\psi\}}, \quad 0 \leq v \leq \psi \quad \text{in } B_{r_1'}$$

and the blowup function v_0 of v at 0 is a halfspace solution. Thus we have the directional monotonicity for v and C^1 regularity of the free boundary $\Gamma(v) = \Gamma^{\psi}(u)$ near 0 by using the same method as that in the above paragraph. $\square$

Chapter 5

Double Obstacle Problem (Fully Nonlinear Case)

5.1 Introduction

Obstacle problems appear in various fields such as fluid filtration in porous media, elstro-plasticity, optimal control and financial mathematics, see [Fri, Caf98]. The regularity of the free boundary in the classical single obstacle problem is first proved by [Caf77]. Thereafter, the existence and uniqueness of the solution, $C^{1,1}$ regularity of the solution, and C^1 (and higher) regularity of the free boundary of various obstacle problems for linear and nonlinear operators have been studied by [CKS, Lee98, Lee01, Iva] and various researchers.

The formulation, existence, uniqueness and $W^{2,p}$ regularity of the solution for double obstacle problem with C^2 obstacles for linear and nonlinear operator was discussed by [MR]. The double obstacle problem with homogeneous degree two polynomial obstacles in two dimension was studied by [Ale]. The regularity of the free boundaries of the double obstacle problem for Laplacian was obtained by [LPS].

In this paper, we prove the existence and uniqueness of $W^{2,p}$ $(n < p < \infty)$ solution of *double obstacle problem* for fully nonlinear operator in a domain $D \subset \mathbb{R}^n$,

$$\begin{cases} F(D^2u, x) \geq 0, & \text{in } \{u > \phi_1\} \cap D, \\ F(D^2u, x) \leq 0, & \text{in } \{u < \phi_2\} \cap D, \\ \phi_1(x) \leq u(x) \leq \phi_2(x) & \text{in } D, \\ u(x) = g(x) & \text{on } \partial D, \end{cases} \tag{FB}$$

with $\phi_1, \phi_2 \in C^{1,1}(\overline{D})$, $\partial D \in C^{2,\alpha}$, $g \in C^{2,\alpha}(\overline{D})$ and $\phi_1 \le g \le \phi_2$ in ∂D. Moreover, we show local C^1 regularity of free boundary for *nosign reduced double obstacle problem*:

$$\begin{aligned} F(D^2u, x) = f\chi_{\Omega(u)\cap\{u<\psi\}} + F(D^2\psi, x)\chi_{\Omega(u)\cap\{u=\psi\}} \quad &\text{a.e. in } B_1, \\ u \le \psi \quad &\text{in } B_1, \end{aligned} \tag{FB_{local}}$$

where

$$\Omega(u) := B_1 \setminus (\{u = 0\} \cap \{\nabla u = 0\}) \quad \text{and} \quad f \in C^{0,1}(B_1),$$

with *the upper obstacle function*

$$\psi \in C^{1,1}(B_1) \cap C^{2,1}(\overline{\Omega(\psi)}), \quad \Omega(\psi) := B_1 \setminus (\{\psi = 0\} \cap \{\nabla\psi = 0\}).$$

For Laplace case, the regularity of two free boundaries of (FB_{local}) with a zero lower obstacle and a upper obstacle $\psi \in C^{1,1}$ whose behavior near zero is of the half-space function type , $\frac{a}{2}(x_n^+)^2$, was discussed by [LPS]. We note that (FB_{local}) is a generalized problem (without the sign condition $0 \le u$) of reduced form of (FB),

$$F(D^2u, x) = f\chi_{\{0<u<\psi\}} + F(D^2\psi, x)\chi_{\{0<u=\psi\}}, \quad 0 \le u \le \psi \text{ in } B_1, \tag{5.1}$$

with $\psi \in C^{1,1}(\overline{B_1}) \cap \psi \in C^{2,1}(\overline{\{\psi > 0\}})$, $f \in C^{0,1}(B_1)$, see Subsection 5.1.1. Hence, we have the regularity of the free boundary near a point on the intersection of two free boundaries $\{\partial u > \phi_1\} \cap \{\partial u < \phi_2\}$, see Corollary 5.1.3. For simplicity, in this introduction, we discuss the ideas of this paper by using the simple form (5.1).

We explain the condition $\psi \in C^{1,1}(\overline{B_1}) \cap C^{2,1}(\overline{\{\psi > 0\}})$. First, we observe that $u = \frac{1}{2}(x_1^+)^2 + \frac{1}{2}(x_2^+)^2$ is a solution of a double obstacle problem, $\Delta u = \chi_{\{0<u<\psi\}} + 2\chi_{\{0<u=\psi\}}$ in $\mathbb{R}^2$, where the upper obstacle function $\psi = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$. Then, the low contact set $\{u = 0\}$ and the upper contact set $\{u = \psi\}$ are the first and third quadrant in $\mathbb{R}^2$, respectively. Hence, the two free boundaries, $\partial\{0 < u < \psi\}$ and $\partial\{0 < u = \psi\}$, have a corner at the origin. It means that if we choose C^2 upper obstacle in (5.1), then the blowup ψ_0 (see Subsection (5.1.4)) of ψ at the free boundary point 0 is a homogeneous degree 2 polynomial. Consequently, it is also possible that free boundaries of the blowup u_0 of u at 0 have a corner. Then, we can not expect more than Lipschitz regularity for the free boundaries.

It presents the reason for choosing $C^{1,1}$ upper obstacle to show the C^1 regularity of the double obstacle problem. Additionally, we assume that $\psi \in C^{2,1}(\overline{\{\psi > 0\}})$ to regard the upper obstacle as a solution of the single obstacle problem. Finally, the thickness assumption for ψ implies that the blowup ψ_0 of ψ at the free boundary point 0 is the half-space function, $\psi_0 = c(x_n^+)^2$ and we have a chance to have the regularity of the free boundaries.

The main difficulty in our work is the lack of monotonicity formulas due to the nonlinearity of the operator, which is the important tools for the analysis on the blowups of solutions at the free boundary point for the double obstacle problem for the Laplace operator, $F(D^2u, x) = \Delta u$, see [LPS]. Hence, we focus on the fact that the blowup ψ_0 of the upper obstacle ψ at 0 is the half-space function, $\psi_0 = c(x_n^+)^2$ and the sign condition $0 \le u_0 \le \psi_0$ implies that $\{u_0 = 0\} \subset \{\psi = 0\} \subset \{x_n \le 0\}$. Thus, we observe that $\partial_e u/x_n$ is finite and prove that the first derivative $\partial_e u_0$ of the blowup u_0 of u in a direction e orthogonal to e_n is identically zero for any direction $e \in \mathbb{S}^{n-1} \cap e_n^\perp$. Finally, u_0 will be characterized as a constant multiple of $(x_n)_+^2$ and we have the regularity of the free boundary of the solution of u, by using the directional monotonicity. It is noticeable that similar arguments for second derivative has been introduced in [LS01] and the one for the first derivative as above has been considered in [IM16b] in different consideration.

Now, we summarize the contents in this paper. In Subsection 5.2.1, we present a modified penalization method which implies the existence and uniqueness of $W^{2,p}$ solution of (FB) with $C^{1,1}$ obstacles. We note that the modified version is simpler than the original one in [Fri, Lee98, Duq]. Furthermore, the condition for obstacle is improved from C^2 to $C^{1,1}$, which is essential for regularity of the free boundaries. We also note that, since $F(D^2u, x)$ is bounded, the $W^{2,p}$ ($n < p < \infty$) solution u of (FB) is the strong L^n solution and the viscosity solution of the problem, see [CCKS]. In Subsection 5.2.2, we prove the optimal regularity of the solution (FB) with $C^{1,1}$ obstacles. In Subsection 5.3.2, we study the analysis on the blowups of solutions at the free boundary point, as discussed in the preceding paragraph. In Subsection 5.3.3, we discuss the directional monotonicity and the proof of the regularity of the free boundaries, Theorem 5.1.2, by using the methods considered in [Lee98, PSU, IM16a, LPS] and references therein.

5.1.1 Reduction of (FB)

By subtracting the lower obstacle ϕ_1, we reduce the problem (FB) to the double obstacle problem with zero lower obstacle, i.e., we define $\tilde{F}(\mathcal{M}, x) := F(\mathcal{M} + D^2\phi_1, x) - F(D^2\phi_1, x)$ and $v = u - \phi_1$, where u is a $W^{2,p}$ $(n < p < \infty)$ solution of (FB). Then

$$\begin{aligned}\tilde{F}(D^2v, x) &= F(D^2u, x) - F(D^2\phi_1, x)\\ &= -F(D^2\phi_1, x)\chi_{\{\phi_1<u<\phi_2\}} + \Big(F(D^2\phi_2, x) - F(D^2\phi_1, x)\Big)\chi_{\{\phi_1<u=\phi_2\}}\\ &= -F(D^2\phi_1, x)\chi_{\{0<v<\phi_2-\phi_1\}} + \tilde{F}(D^2(\phi_2 - \phi_1), x)\chi_{\{0<v=\phi_2-\phi_1\}}.\end{aligned}$$

By replacing $f = -F(D^2\phi_1, x)$, $\psi = \phi_2 - \phi_1$ and reusing $v = u - \phi_1$ by u, $\tilde{F}(\mathcal{M}, x) := F(\mathcal{M} + D^2\phi_1, x) - F(D^2\phi_1, x)$ by $F(\mathcal{M}, x)$, we know that u is a $W^{2,p}$ solution of *reduced double obstacle problem*,

$$F(D^2u, x) = f\chi_{\{0<u<\psi\}} + F(D^2\psi, x)\chi_{\{0<u=\psi\}} \quad \text{a.e. in } D \tag{5.2}$$

with $0 \le u \le \psi$ in D, $f \in L^\infty(D)$ and $\psi \in C^{1,1}(\overline{D})$.

5.1.2 Notations

We will use the following notations throughout the paper.

C, C_0, C_1	generic constants
χ_E	the characteristic function of the set E, $(E \subset \mathbb{R}^n)$
$\overline{E}$	the closure of E
∂E	the boundary of a set E
$\lvert E\rvert$	n-dimensional Lebesgue measure of the set E
$B_r(x), B_r$	$\{y \in \mathbb{R}^n : \lvert y - x\rvert < r\}$, $\quad B_r(0)$
$\Omega(u), \Omega(\psi)$	see Equation (FB_{local})
$\Lambda(u), \Lambda(\psi)$	$B_1 \setminus \Omega(u), B_1 \setminus \Omega(\psi)$
$\Gamma(u), \Gamma^\psi(u)$	$\partial\Lambda(u) \cap B_1, \{u = \psi\} \cap B_1$
$\Gamma^d(u)$	$\Gamma(u) \cap \Gamma^\psi(u)$ (the intersection of free boundaries)
$\partial_\nu, \partial_{\nu e}$	first and second directional derivatives
$P_r(M), P_\infty(M)$	see Definition 5.1.2, 5.1.3
$\delta_r(u, x), \delta_r(u)$	see Definition 5.1.1
$\mathcal{P}^+, \mathcal{P}^-$	Pucci operators
$\mathcal{S}, \overline{\mathcal{S}}, \underline{\mathcal{S}}, \mathcal{S}^*$	the viscosity solution spaces for the Pucci operators

5.1.3 Conditions on $F = F(\mathcal{M}, x)$

For the definitions of the viscosity solution and the spaces of viscosity solutions of the Pucci operators $\mathcal{S}(\lambda_0, \lambda_1, f)$, $\overline{\mathcal{S}}(\lambda_0, \lambda_1, f)$, $\underline{\mathcal{S}}(\lambda_0, \lambda_1, f)$, $\mathcal{S}^*(\lambda_0, \lambda_1, f)$, we refer to the book of Caffarelli-Cabré [CC]. We assume that the fully nonlinear operator $F(\mathcal{M}, x)$ satisfies the following conditions:

(F1) $F(0, x) = 0 \quad$ for all $x \in \mathbb{R}^n$.

(F2) F is uniformly elliptic with ellipticity constants $0 < \lambda_0 \leq \lambda_1 < +\infty$, that is

$$\lambda_0 \|\mathcal{N}\| \leq F(\mathcal{M} + \mathcal{N}, x) - F(\mathcal{M}, x) \leq \lambda_1 \|\mathcal{N}\|,$$

for any symmetric $n \times n$ matrix $\mathcal{M}$ and positive definite symmetric $n \times n$ matrix $\mathcal{N}$.

(F3) $F(\mathcal{M}, x)$ is convex in $\mathcal{M}$ for all $x \in \mathbb{R}^n$.

(F4)

$$|F(\mathcal{M}, x) - F(\mathcal{M}, y)| \leq C|\mathcal{M}||x - y|.$$

Furthermore, we introduce the *Pucci operators* $\mathcal{P}^{\pm}$, with $0 < \lambda_0 \leq \lambda_1 < +\infty$ as

$$\mathcal{P}^-(\mathcal{M}, \lambda_0, \lambda_1) := \inf_{\lambda_0 Id \leq \mathcal{N} \leq \lambda_1 Id} Tr \mathcal{N}\mathcal{M}, \quad \mathcal{P}^+(\mathcal{M}, \lambda_0, \lambda_1) := \sup_{\lambda_0 Id \leq \mathcal{N} \leq \lambda_1 Id} Tr \mathcal{N}\mathcal{M},$$

for any symmetric $n \times n$ matrix $\mathcal{M}$.

5.1.4 Definitions

In order to find the possible configuration of the solution near the free boundary, the following blowup concept has been used heavily at [Caf77, Fri] and other references.

For a $W^{2,n}$ solution, u, of (FB_{local}) in B_r, we define the *rescaling function* of u at x_0 with $\rho > 0$ as

$$u_\rho(x) = u_{\rho, x_0}(x) := \frac{u(x_0 + \rho x) - u(x_0)}{\rho^2}, \quad x \in B_{r/\rho}.$$

By optimal $C^{1,1}$ regularity of solution u (Theorem 5.1.1), for any sequence $\rho_i \to 0$, there exists a subsequence ρ_{i_j} of ρ_i and $u_0 \in C^{1,1}_{loc}(\mathbb{R}^n)$ such that

$$u_{\rho_{i_j}} \to u_0 \text{ uniformly in } C^{1,\alpha}_{loc}(\mathbb{R}^n) \quad \text{for any } 0 < \alpha < 1.$$

The limit function u_0 is a *blowup of u at x_0*.

To measure the contact set $\Lambda(u)$ between u and zero lower obstacle, we are going to define the minimal distance of contact sets as [Caf77].

Definition 5.1.1. We denote by $\delta_r(u,x)$ the thickness of $\Lambda(u)$ on $B_r(x)$, i.e.,

$$\delta_r(u,x) := \frac{\mathrm{MD}(\Lambda(u)\cap B_r(x))}{r},$$

where $\mathrm{MD}(A)$ is the least distance between two parallel hyperplanes containing A. We will use the abbreviated notation $\delta_r(u)$ for $\delta_r(u,0)$.

In order to state a theorem on the regularity of free boundary, we define classes of local and global solutions of the problem.

Definition 5.1.2. (Local solutions) We say a $W^{2,n}$ function u belongs to the class $P_r(M)$ $(0<r<\infty)$, if u satisfies :

(i) $F(D^2u,x) = f\chi_{\Omega(u)\cap\{u<\psi\}} + F(D^2\psi,x)\chi_{\Omega(u)\cap\{u=\psi\}}, \quad u \le \psi \quad$ in B_r,

(ii) $\|D^2u\|_{\infty,B_r} \le M$,

(iii) $0 \in \Gamma^d(u)$,

where $f \in C^{0,1}(B_r)$ and $\psi \in C^{1,1}(B_r)\cap C^{2,1}(\overline{\Omega(\psi)})$.

Definition 5.1.3. (Global solutions) We say a $W^{2,n}$ function u belongs to the class $P_\infty(M)$, if u satisfies :

(i) $F(D^2u) = \chi_{\Omega(u)\cap\{u<\psi\}} + F(D^2\psi)\chi_{\Omega(u)\cap\{u=\psi\}}, \quad u \le \psi \quad$ in $\mathbb{R}^n$,

(ii) $F(D^2\psi) = a\chi_{\Omega(\psi)}$ in $\mathbb{R}^n$, for a constant $a>1$,

(iii) $\|D^2u\|_{\infty,\mathbb{R}^n} \le M$,

(iv) $0 \in \Gamma(u)$.

5.1.5 Main Theorems

The purpose of the paper is to obtain the existence, optimal regularity of the solution for the double obstacle problem and the regularity of the free boundary. The main theorems are as follows:

Theorem 5.1.1 (Existence and optimal regularity)**.** *Assume F satisfies (F1)-(F4). Then the following holds:*

(i) There exist $W^{2,n}$ solution u of (FB) with $\phi_1, \phi_2 \in C^{1,1}(\overline{D})$, $\partial D \in C^{2,\alpha}$, $g \in C^{2,\alpha}(\overline{D})$ and $\phi_1 \leq g \leq \phi_2$ in D.

(ii) For any compact set K in D, we have

$$\|u\|_{C^{1,1}(K)} \leq M < \infty,$$

for some constant $M = M(\|u\|_{L^\infty(D)}, \|\phi_1\|_{C^{1,1}(D)}, \|\phi_2\|_{C^{1,1}(D)}, dist(K, \partial D)) > 0$.

Theorem 5.1.2 (Regularity of free boundary)**.** *Assume F satisfies (F1)-(F4) and F is C^1 and let $u \in P_1(M)$ with an upper obstacle ψ such that*

$$0 \in \partial\Omega(\psi), \qquad \lim_{x \to 0, x \in \Omega(\psi)} F(D^2\psi(x), x) > f(0), \quad f \geq c_0 > 0 \text{ in } B_1,$$

and

$$\inf\left\{F(D^2\psi, x), F(D^2\psi, x) - f\right\} \geq c_0 > 0 \text{ in } \Omega(\psi).$$

Suppose

$$\delta_r(\psi, z) := \frac{MD(\Lambda(u) \cap \Lambda(\psi) \cap B_r)}{r} \geq \epsilon_0 \quad \text{for all } r < 1/4, z \in \Gamma(u) \tag{5.3}$$

and

$$\delta_r(u, \psi) \geq \epsilon_0 \quad \text{for all } r < 1/4. \tag{5.4}$$

Then there is $r_0 = r_0(u, c_0, \|\nabla F\|_{L^\infty(B_M \times B_1)}, \|F\|_{L^\infty(B_M \times B_1)}, \|\nabla f\|_{L^\infty(B_1)}) > 0$ such that $\Gamma(u) \cap B_{r_0}$ is a C^1 graph.

Corollary 5.1.3. *If we assume further that $\phi_2 - \phi_1 \in C^{2,1}(\overline{\{\phi_2 - \phi_1 > 0\}})$, $\phi_1 \in C^{2,1}(D)$, then after translation and rescaling the solution of (FB) is in $P_1(M)$, see Subsection 5.1.1. Hence with additional assumption corresponding those in Theorem 5.1.2, we have the local C^1 regularity of the free boundary near the free boundary point on $\{\partial u > \phi_1\} \cap \{\partial u < \phi_2\}$.*

The thickness assumption (5.3) and (5.4) in Theorem 5.1.2 implies

$$\delta_r(\psi_0, x_0) \geq \epsilon_0 \quad \text{for all } r > 0, x_0 \in \Gamma(\psi_0)$$

and

$$\delta_r(u_0, \psi_0) \geq \epsilon_0 \quad \text{for all } r > 0,$$

for any blowups u_0 and ψ_0 of u and ψ at 0, respectively. The uniform thickness assumption for ψ, (5.3) is to ensure that the blowup ψ_0 of ψ is the half-space type upper obstacle, $\psi_0 = \frac{a}{2}(x_n^+)^2$. The thickness assumption for $\Lambda(u) \cap \Lambda(\psi)$ enables us to have the regularity of the free boundary without the sign condition, $u \geq 0$.

5.2 Existence, Uniqueness and Optimal Regularity

5.2.1 Existence, uniqueness of $W^{2,p}$ solution

In this subsection, we suggest a modified penalization method. The main difference from the original one in [Fri, Lee98, Duq] and our modified method is uniform boundedness of the penalization term β_ϵ. As a result, the proof of the modified version is simper than that of the original version and the condition for obstacles is improved from C^2 to $C^{1,1}$. We note that $C^{1,1}$ condition for the obstacles is demanded to have the regularity of the free boundary, see the Introduction of this paper.

Proposition 5.2.1. *Assume F satisfies (F1)-(F4). There is a unique solution $u \in W^{2,p}(D)$, for any $n < p < \infty$, of (FB) with*

$$\|u\|_{W^{2,p}(D)} \leq C\left(\left\|F(D^2\phi_1, x)\right\|_{L^\infty(D)}, \left\|F(D^2\phi_2, x)\right\|_{L^\infty(D)}\right),$$

where $\phi_1, \phi_2 \in C^{1,1}(\overline{D})$, $\partial D \in C^{2,\alpha}$, $g^{2,\alpha} \in C(\overline{D})$ and $\phi_1 \leq g \leq \phi_2$ on ∂D.

Proof. Let $\beta_1(z) \in C^\infty(\mathbb{R})$ be a function satisfying

$$\begin{cases} \beta_1(z) = -\max\left\{\left\|F(D^2\phi_1, x)\right\|_{L^\infty(D)}, \left\|F(D^2\phi_2, x)\right\|_{L^\infty(D)}\right\} & \text{if } z < -1, \\ \beta_1(z) = 0 & \text{if } z > 1, \\ \beta_1(z) \leq 0 & \text{in } z \in \mathbb{R}, \end{cases}$$

and define $\beta_\epsilon(z) := \beta_1(z/\epsilon)$, for $\epsilon > 0$. We consider the penalization problem,

$$\begin{cases} F(D^2u, x) = \beta_\epsilon(u - \phi_1) - \beta_\epsilon(\phi_2 - u) & \text{in } D, \\ u(x) = g(x) & \text{on } \partial D. \end{cases} \tag{5.5}$$

By the $W^{2,p}$ regularity in [CC], for each $v \in C^{0,\alpha}(D)(0 < \alpha < 1)$ there is a unique solution $w \in W^{2,p}(D)$, for any $n < p < \infty$, of

$$\begin{cases} F(D^2 w, x) = \beta_\epsilon(v - \phi_1) - \beta_\epsilon(\phi_2 - v) & \text{in } D, \\ w(x) = g(x) & \text{on } \partial D. \end{cases}$$

Since the boundedness of β_ϵ,

$$||w||_{W^{2,p}(D)} \leq C_0,$$

where C_0 is a constant which is independent for ϵ and v.

Let $w = Sv$, then $S : B_{C_0} \to B_{C_0}$ is a compact map since $W^{2,p}$ compactly embeded in $C^{0,\alpha}$, where B_{C_0} is the C_0 ball centered at 0 in $C^{0,\alpha}(D)$. By C^α estimate up to boundary for solutions of inhomogeneous Dirichlet problem for fully nonlinear operator, we know that S is continuous. By Schauder's fixed-point theorem, there is a function $u \in B_{C_0}$ such that $Su = u$. That means there is $u_\epsilon \in W^{2,p}(D)$ such that u_ϵ is a solution of (5.5) and $||u_\epsilon||_{W^{2,p}} \leq C_0$, where C_0 is not depending ϵ. Then there is a sequence $\epsilon = \epsilon_i \to 0$ and $u \in W^{2,p}(D)$ such that

$$u_\epsilon \to u \quad \text{weakly in } W^{2,p}(D), \quad \text{for all } n < p < \infty.$$

Thus, we know that $||u||_{W^{2,p}(D)} \leq C_0$ and

$$u_\epsilon \to u \quad \text{uniformly in } D.$$

We claim that u is a solution of the double obstacle problem, (FB). First, we are going to prove that $F(D^2u, x) \geq 0$ in $\{u > \phi_1\} \cap D$. Take a point x_0 in $\{u > \phi_1\} \cap D$ and let $\delta = (u(x_0) - \phi_1(x_0))/2$. Then, by the uniform convergence of u_ϵ to u, there is a ball $B_r(x_0) \Subset \{u > \phi_1\} \cap D$ and $\epsilon_0 > 0$ such that $u_\epsilon - \phi_1 \geq \delta$ in $B_r(x_0)$, for $\epsilon < \epsilon_0$. By the definition of β_ϵ, for $\epsilon \leq \min\{\epsilon_0, \delta\}$, we have

$$\beta_\epsilon(u_\epsilon - \phi_1) \equiv 0 \quad \text{and} \quad F(D^2 u_\epsilon, x) \geq 0 \text{ in } B_r(x_0).$$

By the closedness of the family of viscosity solutions, Proposition 2.9 of [CC], the uniform convergence of u_ϵ to u implies that $F(D^2u, x) \geq 0$ in $B_r(x_0)$. Since $x_0 \in \{u > \phi_1\} \cap D$ is arbitrary, we obtain $F(D^2u, x) \geq 0$ in $\{u > \phi_1\} \cap D$. We also have $F(D^2u, x) \leq 0$ in $\{u < \phi_2\} \cap D$, by the same method.

Next, we are going to prove $\phi_1 \le u \le \phi_2$ in D. Suppose that $\{u < \phi_1\} \cap D$ is not empty and take a point $x_0 \in \{u < \phi_1\} \cap D$. Then, by uniform convergence of u_ϵ, there is a ball $B_r(x_0)$ such that

$$\beta_\epsilon(u_\epsilon - \phi_1) = -\max\left\{\left\|F(D^2\phi_1, x)\right\|_{L^\infty(D)}, \left\|F(D^2\phi_2, x)\right\|_{L^\infty(D)}\right\}, \beta_\epsilon(\phi_2 - u_\epsilon) \equiv 0 \text{ in } B_r(x_0)$$

and

$$F(D^2u_\epsilon, x) \le F(D^2\phi_1) \text{ in } B_r(x_0), \text{ for sufficiently small } \epsilon.$$

Consequently, $F(D^2u, x) \le F(D^2\phi_1)$ in $\{u < \phi_1\} \cap D$. Moreover, the boundary condition implies that $u \equiv \phi_1$ on $\partial(\{u < \phi_1\} \cap D)$. Hence, by the maximum principle, $u \ge \phi_1$ in $\{u < \phi_1\} \cap D$ and it is a contradiction. The same method implies that $\{u > \phi_2\} \cap D = \emptyset$ and $\phi_1 \le u \le \phi_2$ in D.

Suppose that there are two solutions u_1 and u_2 of (FB) and $\{u_1 < u_2\} \cap D$ is not empty. Then $\phi_2 \ge u_2 > u_1$ and $u_2 > u_1 \ge \phi_1$ in $\{u_1 < u_2\} \cap D$ and $F(D^2u_1, x) \le 0 \le F(D^2u_2, x)$ in $\{u_1 < u_2\} \cap D$. The boundary condition implies that $u_1 \equiv u_2$ on $\partial(\{u_1 < u_2\} \subset D)$ and we have $u_1 \ge u_2$ in $\{u_1 < u_2\} \subset D$, by comparison principle. Therefore, we arrive at a contradiction and have the uniqueness of the solution of (FB). □

5.2.2 Optimal Regularity

In this subsection, we prove the optimal regularity of the double obstacle problem (FB) with $C^{1,1}$ obstacles, by using the reduced form of (FB) explained in Subsection 5.1.1. We start with a definition of a solution space.

Definition 5.2.1. For a positive constant c', let $\mathcal{G}(c')$ be a class of solution $u \in W^{2,n}(B_1)$ of

$$F(D^2u, x) = f(x)\chi_{\{0<u<\phi\}} + F(D^2\phi, x)\chi_{\{0<u=\phi\}}, \quad 0 \le u \le \phi \text{ in } B_1. \tag{5.6}$$

with $|f(x)|, |F(D^2\phi, x)|, |\phi| \le c'$ in B_1 and $0 \in \Gamma(u)$.

Proposition 5.2.2 (Quadratic growth)**.** *Assume F satisfies (F1), (F2). For $u \in \mathcal{G}(c')$, we have*

$$S(r, u) = \sup_{x \in B_r} u(x) \le C_0 r^2,$$

for a positive constant $C_0 = C_0(c')$.

Proof. First, we are going to show that there is a positive constant C_0 such that

$$S(2^{-j-1}, u) \leq max(C_0 2^{-2j}, 2^{-2} S(2^{-j}, u)) \quad \text{for all } j \in \mathbb{N} \cup \{0, -1\}. \tag{5.7}$$

Suppose it fails. Then, for each $j \in \mathbb{N} \cup \{0, -1\}$, there exists $u_j \in \mathcal{G}$ such that

$$S(2^{-j-1}, u_j) > max(j 2^{-2j}, 2^{-2} S(2^{-j}, u_j)). \tag{5.8}$$

We define

$$\tilde{u}_j(x) := \frac{u(2^{-j}x)}{S(2^{-j-1}, u)} \quad x \in B_{2^j}.$$

Then, by the definition of $\tilde{u}$ and (5.8),

$$S(\tilde{u}_j, 1/2) = 1, \quad S(\tilde{u}_j, 1) = 4, \quad \tilde{u}_j(0) = 0.$$

Since $u \in \mathcal{G}(c')$, by the condition (F1) and Proposition 2.13 of [CC], we know that $u \in \mathcal{S}^*(\frac{\lambda}{n}, \Lambda, c')$. By (5.8),

$$\mathcal{P}^+(D^2\tilde{u}(x)) = \frac{2^{-2j}}{S(2^{-j-1}, u)} \cdot \mathcal{P}^+(D^2 u(2^{-j}x)) \geq -\frac{c'}{j}$$

and

$$\mathcal{P}^-(D^2\tilde{u}(x)) = \frac{2^{-2j}}{S(2^{-j-1}, u)} \cdot \mathcal{P}^-(D^2 u(2^{-j}x)) \leq \frac{c'}{j},$$

i.e., $\tilde{u} \in \mathcal{S}^*(\lambda/n, \Lambda, c'/j)$. By Harnack inequality (Theorem 4.3 of [CC]) and C^α regularity (Proposition 4.10 of [CC]), we know that $\tilde{u}_j \to \tilde{u}$ in B_1, up to subsequence and

$$\tilde{u} \in \mathcal{S}^*(\lambda/n, \Lambda, 0) \quad \text{in } B_1,$$

$\tilde{u} \neq 0$ in $B_{1/2}$ and $\tilde{u}(0) = 0$. Then, by strong maximum principle, we have a contradiction.

Next, we claim that

$$S(2^{-j}, u) \leq C_0 2^{-2j+2} \quad \text{for all } j \in \mathbb{N} \cup \{0\}. \tag{5.9}$$

We may assume that $C_0 > c'/4$. Then (5.9) holds for $j = 0$. Assume that (5.9) holds for $j = j_0$. Then, by (5.7),

$$S(2^{-(j_0+1)}, u) \leq max(C_0 2^{-2j_0}, 2^{-2} S(2^{-j_0}, u)) \leq C_0 2^{-2j_0},$$

i.e., by the induction, we have (5.9) for all $j \in \mathbb{N} \cup \{0\}$.

Let $2^{-j-1} \leq r \leq 2^{-j}$. Then, by (5.9),

$$S(r,u) \leq S(2^{-j},u) \leq C_0 2^{-2j+2} = C_0 2^4 2^{-2j-2} \leq C_0 2^4 r^2.$$

Thus, we have the quadratic growth rate of for u at 0. $\square$

Proposition 5.2.3 (Optimal regularity)**.** *Assume F satisfies (F1)-(F4). Let $u \in W^{2,n}(D)$ be a solution of (FB), with $\phi_1, \phi_2 \in C^{1,1}(\overline{D})$, $\partial D \in C^{2,\alpha}$, $g \in C^{2,\alpha}(\overline{D})$ and $\phi_1 \leq g \leq \phi_2$ on ∂D. Then $u \in W^{2,\infty}_{loc}(D)$.*

Proof. Let K be a compact set in D and $\delta = dist(K, \partial D)$. Since $u \in W^{2,p}(D)$, $D^2 u = D^2\phi_1$ a.e. on $\{u = \phi_1\}$ and $D^2 u = D^2\phi_2$ a.e. on $\{u = \phi_2\}$. Thus it suffice to show that $\|u\|_{W^{2,\infty}(\{\phi_1 < u < \phi_2\} \cap K)} < +\infty$. Let

$$x_0 \in \{\phi_1 < u < \phi_2\} \cap K$$

and $d(x_0) = dist(x_0, \partial\{u = \phi_1\} \cup \partial\{u = \phi_2\})$.

We are going to apply Proposition 5.2.2 to $u - \phi_1$ and $\phi_2 - u$ which are solutions of the problem of the form (5.6) (see e.g. the introduction of our paper and Remark 5.3.1).

First, we assume that $d(x_0) = dist(x_0, \partial\{u = \phi_1\} \cup \partial\{u = \phi_2\}) = dist(x_0, \partial\{u = \phi_1\})$.

Case 1) $5d(x_0) < \delta$

Let $y_0 \in \partial B_{d(x_0)}(x_0) \cap \{u = \phi_1\}$. Then $B_{4d}(y_0) \subset B_{5d}(x_0) \Subset D$. By Proposition 5.2.2 for $u(4dx + y^0) - \phi_1(4dx + y^0)/(4d)^2$, we obtain

$$\|u - \phi_1\|_{L^\infty(B_{2d}(y_0))} \leq C(\|\phi_1\|_{C^{1,1}(\overline{D})}, \|\phi_2\|_{C^{1,1}(\overline{D})}) d^2.$$

Since $\tilde{F}(D^2(u - \phi_1), x) = -F(D^2\phi_1, x)$ in $\{\phi_1 < u < \phi_2\}$, where $\tilde{F}(\mathcal{M}, x) = F(\mathcal{M} + D^2\phi_1, x) - F(D^2\phi_1, x)$, by interior estimate,

$$\left\|D^2(u - \phi_1)\right\|_{L^\infty(B_{d/2}(x_0))} \leq C \frac{\|u - \phi_1\|_{L^\infty(B_d(x_0))}}{d^2}.$$

Since $B_d(x_0) \subset B_{2d}(y_0)$,

$$\left\|D^2(u - \phi_1)\right\|_{L^\infty(B_{d/2}(x_0))} \leq C(\|\phi_1\|_{C^{1,1}(\overline{D})}, \|\phi_2\|_{C^{1,1}(\overline{D})}).$$

Case 2) $5d(x_0) > \delta$

The interior derivative estimate for u in $B_{\delta/4}(x_0)$ gives

$$\left\|D^2(u - \phi_1)\right\|_{L^\infty\left(B_{\delta/10}(x_0)\right)} \leq C \frac{4^2}{\delta^2} \|u - \phi_1\|_{L^\infty(D)}.$$

By the regularity for ϕ_1, we have the optimal regularity.

For the case $d(x_0) = dist(x_0, \partial\{u = \phi_2\})$, the same argument as above shows the boundedness of the second derivative of u. □

5.3 Regularity of the Free Boundary

5.3.1 Non-degeneracy

In this section we present the non-degenracy of the solution $u \in P_1(M)$. The non-degeneracy implies that $o \in \Gamma(u_0)$, where u_0 is a blow-up of u at 0 and that $\Gamma(u)$ has a Lebesgue measure zero. The following non-degeneracy estimate comes from a simple barrier argument.

Lemma 5.3.1. *Assume F satisfies (F1), (F2). Let $u \in P_1(M)$. If $f \geq c_0 > 0$ in B_1 and $F(D^2\psi, x) \geq c_0 > 0$ in $\Omega(\psi)$, then*

$$\sup_{\partial B_r(x)} u \geq u(x) + \frac{c}{8\lambda_1 n} r^2 \quad x \in \overline{\Omega(u)} \cap B_1,$$

where $B_r(x) \Subset B_1$.

Proof. Let $x^0 \in \Omega(u) \cap B_1$ and $u(x^0) > 0$. Since $\Omega(u) \cap \{u = \psi\} \subset \Omega(\psi)$ and the assumptions for f and $F(D^2\psi, x)$, we obtain $F(D^2u, x) = f\chi_{\Omega(u)\cap\{u<\psi\}} + F(D^2\psi, x)\chi_{\Omega(u)\cap\{u=\psi\}} \geq c_0$ in $\Omega(u)$. Thus, by the uniformly ellipticity, (F2) in Definition 5.1.3, we have

$$F(D^2\phi, x) \geq F(D^2u, x) - c \geq 0 \text{ on } B_r(x^0) \cap \Omega(u),$$

where

$$\phi(x) := u(x) - u(x^0) - \frac{c}{2\lambda_1 n}|x - x^0|^2.$$

Then, by using maximum principle on $B_r(x^0) \cap \Omega(u)$, $\phi(x^0) = 0$ and $\phi(x) < 0$ on $\partial\Omega(u)$, we have

$$\sup_{\partial B_r(x^0)\cap\Omega(u)} \phi > 0 \quad \text{and} \quad \sup_{\partial B_r(x)} u \geq u(x) + \frac{c}{2\lambda_1 n} r^2.$$

We omit the rest of the proof since it is a repetition of the arguments for the linear case in the proof of Lemma 2.2 of [LPS]. □

Remark. For $u \in P_1(M)$, the non-degeneracy implies that $\sup_{\partial B_s} u_0 \geq \frac{c}{8\lambda_1 n} s^2$ and $0 \in \Gamma(u_0)$. Moreover, if we assume that u is a non-negative function, then $v := \psi - u$ is a solution of

$$\tilde{F}(D^2 v, x) = \Big(F(D^2\psi, x) - f\Big)\chi_{\{0<v<\psi\}} + F(D^2\psi, x)\chi_{\{0<v=\psi\}} \quad \text{in } B_1,$$

where $\tilde{F}(M, x) = F(D^2\psi, x) - F(D^2\psi - M, x)$. Then we have the non-degeneracy for v, $0 \in \Gamma^d(u_0)$ and $|\Gamma^\psi(u)| = 0$, although v is not in $P_1(M)$.

We also note that

$$\tilde{F}(M, x) = F(D^2\psi, x) - F(D^2\psi - M, x)$$

is a concave fully nonlinear operator. Thus, we can not apply the theory of the obstacle problem for convex fully nonlinear operator in [Lee98] to $\tilde{F}(M, x)$. Precisely, we can not have Lemma 5.3.7 for v, which is that the blowup of v at $x \in \Gamma(v) = \Gamma^\psi(u)$ near 0 is a half-space solution, $v = c(x_n^+)^2$. Hence, in contrast with linear theory [LPS], we only have the regularity of the free boundary $\Gamma(u)$, in this paper.

5.3.2 Classification of Global Solutions

For the global solution $u \in P_\infty(M)$ with the upper obstacle $\psi = \frac{a}{2}(x_n^+)^2$, we suppose the thickness assumption $\delta_r(u, \psi) \geq 0$, for all $r > 0$. Then, the non-degeneracy implies that $\{x_n < 0\} \subset \{u = 0\} = \Lambda(u)$. Thus, we observe that $\partial_e u / x_n$ is finite and prove that $\partial_e u \equiv 0$ for any direction $e \in \mathbb{S}^{n-1} \cap e_n^\perp$. It implies that u is a *half-space solution*, $u = c(x_n^+)^2$.

Proposition 5.3.2. *Assume F satisfies (F1)-(F4). Let $u \in P_\infty(M)$ be a solution of*

$$F(D^2 u) = \chi_{\Omega(u)\cap\{u<\psi\}} + a\chi_{\Omega(u)\cap\{u=\psi\}}, \quad u \leq \psi \quad \textit{a.e. in } \mathbb{R}^n,$$

with the upper obstacle

$$\psi(x) = \frac{a}{2}(x_n^+)^2,$$

for a constant $a > 1$. Suppose

$$\delta_r(u, \psi) \geq 0 \quad \textit{for all } r > 0.$$

Then we have

$$u(x) = \frac{1}{2}(x_n^+)^2 \quad \textit{or} \quad u(x) = \frac{a}{2}(x_n^+)^2.$$

Proof. By the condition $u \le \psi = \frac{a}{2}(x_n^+)^2$ on $\mathbb{R}^n$, we know that $u(x) \le 0$ on $\{x_n \le 0\}$. Suppose that $\partial\Omega(u)\cap\{x_n < 0\} \ne 0$. Then, by non-degeneracy, (Lemma 5.3.1), we know that $\{u > 0\} \cap \{x_n < 0\} \ne 0$ and we arrive at a contradiction. Next, suppose that $\{x_n < 0\} \subset \Omega(u)$. Since $\{\psi = 0\} = \Lambda(\psi) = \{x_n \le 0\}$, it is a contraction to $\delta_r(u \cap \psi) \ge 0$, for all $r > 0$. Thus, the only possibility is $\{x_n < 0\} \subset \Lambda(u)$. In other words, we have

$$\Omega(u) \subset \{x_n > 0\}$$

and $u = 0$ on $\{x_n \le 0\}$ and $\partial_e u = 0$ on $\{x_n \le 0\}$ for all $e \in \mathbb{S}^{n-1} \cap e_n^\perp$, where $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$ and $e^\perp := \{x \in \mathbb{R}^n : x \perp e\}$ for $e \in \mathbb{S}^{n-1}$.

It suffice to show that $\partial_e u \equiv 0$ on $\mathbb{R}^n$ for any direction $e \in \mathbb{S}^{n-1} \cap e_n^\perp$ to have the conclusion. Then, we fix $e_1 \in \mathbb{S}^{n-1} \cap e_n^\perp$ and define

$$0 \le \sup_{x\in\{x_n>0\}} \frac{\partial_1 u(x)}{x_n} =: M_0.$$

M_0 is finite since the optimal regularity and $\partial_1 u = 0$ on $\{x_n \le 0\}$. Arguing by contradiction, suppose $M_0 > 0$.

Since $\partial_1 u \equiv 0$ on $(\Omega(u) \cap \{u < \psi\})^c$, we can take a sequence $x^j \in \Omega(u) \cap \{u < \psi\} \subset \{x_n > 0\}$ such that

$$\lim_{j\to\infty} \frac{1}{x_n^j}\partial_1 u(x^j) = M_0.$$

Let $r_j := x_n^j = |x_n^j|$ and consider rescaling functions

$$u_{r_j}(x) := \frac{u\left(((x^j)', 0) + r_j x\right)}{(r_j)^2} \quad \text{and} \quad \psi_{r_j}(x) := \frac{\psi\left(((x^j)', 0) + r_j x\right)}{(r_j)^2} = \psi(x).$$

Then, $D^2 u_{r_j}$ are uniformly bounded and $u_{r_j} \equiv 0$ on $\{x_n \le 0\}$. Thus,

$$u_{r_j}(x) \to \tilde{u}(x) \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^n) \text{ for any } \alpha \in [0, 1),$$

$$\tilde{u} \equiv 0 \quad \text{on } \{x_n \le 0\}\,(\Omega(\tilde{u}) \subset \{x_n > 0\}) \tag{5.10}$$

and $\tilde{u}$ is a solution of

$$F(D^2u) = \chi_{\Omega(u)\cap\{u<\psi\}} + a\chi_{\Omega(u)\cap\{u=\psi\}}, \quad u \le \psi \quad \text{a.e. in } \mathbb{R}^n,$$

with the upper obstacle

$$\psi(x) = \frac{a}{2}(x_n^+)^2.$$

By the definition of M, for $x \in \{x_n > 0\}$,

$$\partial_1 u_{r_j}(x) = \frac{\partial_1 u\Big(((x^j)', 0) + r_j x\Big)}{r_j x_n} \cdot x_n \le M_0 x_n.$$

Hence, we have $\partial_1 \tilde{u}(x) \le M_0 x_n$ on $\{x_n > 0\}$. Moreover,

$$\partial_1 \tilde{u}(e_n) = \lim_{j\to\infty} \partial_1 u_{r_j}(e_n) = \lim_{j\to\infty} \frac{\partial_1 u\Big(((x^j)', 0) + r_j e_n\Big)}{r_j} = \lim_{j\to\infty} \frac{\partial_1 u(x^j)}{x_n^j} = M_0.$$

If $e_n \in (\Omega(\tilde{u}) \cap \{\tilde{u} < \psi\})^c$, then $\partial_1 \tilde{u}(e_n) = 0$ and we arrive at a contradiction. Thus, $e_n \in \Omega(\tilde{u}) \cap \{\tilde{u} < \psi\}$. Let $\tilde{\Omega}(\tilde{u})$ be the connected component of $\Omega(\tilde{u}) \cap \{\tilde{u} < \psi\}$ containing e_n. By (5.10), we know that $\tilde{\Omega}(\tilde{u}) \subset \Omega(\tilde{u}) \subset \{x_n > 0\}$.

By differentiating $F(D^2\tilde{u}) = 1$ on $\tilde{\Omega}(\tilde{u})$, we have $F_{ij}(D^2\tilde{u})\partial_{ij}\partial_1\tilde{u} = F_{ij}(D^2\tilde{u})$ $\partial_{ij} M_0 x_n = 0$ on $\tilde{\Omega}(\tilde{u})$. Thus, the strong maximum principle implies that

$$\partial_1 \tilde{u} = M_0 x_n \quad \text{ on } \tilde{\Omega}(\tilde{u}) \subset \{x_n > 0\}.$$

If there exists $x \in \partial\tilde{\Omega}(\tilde{u}) \cap \{x_n > 0\}$, then $\partial_1 \tilde{u}(x) = 0 = M x_n$ and we have a contradiction, i.e., we have $\partial\tilde{\Omega}(\tilde{u}) \cap \{x_n > 0\} = \emptyset$. Then, $\tilde{\Omega}(\tilde{u}) \subset \{x_n > 0\}$ implies $\tilde{\Omega}(\tilde{u}) = \{x_n > 0\}$, $\partial_1\tilde{u} \equiv M x_n$ on $\{x_n > 0\}$ and

$$\tilde{u}(x) = M_0 x_1 x_n + g(x_2, ...x_n) \quad \text{ on } \{x_n > 0\}.$$

Since $\tilde{u} = 0$ and $\nabla\tilde{u} = 0$ on $\{x_n = 0\}$, by using the even extension function of $\tilde{u}$ and the Evans-Krylov theorem, we have $C^{2,\alpha}$-estimate for $\tilde{u}(Rx)/R^2$ up to boundary of $B_1 \cap \{x_n > 0\}$. Thus, we obtain

$$\sup_{x,y\in B_R\cap\{x_n>0\}} \frac{|D^2\tilde{u}(x) - D^2\tilde{u}(y)|}{|x-y|^\alpha} \le \frac{C}{R^\alpha}$$

and letting $R \to \infty$ implies that $D^2\tilde{u}$ is a constant in $\{x_n > 0\}$. Then $\tilde{u}$ is a second order polynomial in $\{x_n > 0\}$ with $\tilde{u} = 0$ on $\{x_n \le 0\}$, $\nabla\tilde{u} = 0$ on $\{x_n \le 0\}$. Since $\tilde{u} \in C^{1,\alpha}_{loc}(\mathbb{R}^n)$, $\tilde{u} = \frac{1}{2}x_n^2$ on $\{x_n > 0\}$ and it is a contradiction to $M_0 > 0$. □

5.3.3 Directional Monotonicity and proof of Theorem 5.1.2

In this subsection, we show the directional monotonicity for solutions to (FB_{local}) and the regularity of the solutions of problem. We note that the argument for linear case is discussed in [LPS] and for the single obstacle problem for fully nonlinear operator explained in [Lee98, IM16a].

Lemma 5.3.3. *Assume F satisfies (F1)-(F4) and F is C^1 and let u be a solution of*

$$F(D^2u, rx) = f(rx)\chi_{\Omega(u)\cap\{u<\psi\}} + F(D^2\psi(rx), rx)\chi_{\Omega(u)\cap\{0<u=\psi\}}, \quad u \le \psi \quad \text{in } B_1$$

and assume that $f(x) \ge c_0 > 0$ in B_1, Suppose that we have

$$C\partial_e\psi - \psi \ge 0, \quad C\partial_e u - u \ge -\epsilon_0 \quad \text{in } B_1,$$

for a direction e and $\epsilon_0 < c/64\lambda_1 n$. Then we obtain

$$C\partial_e u - u \ge 0 \quad \text{in } B_{1/2},$$

if $0 < r \le r_0'$, for some

$$r_0' = r_0'(C, c_0, \|\nabla F\|_{L^\infty(B_M\times B_1)}, \|\nabla f\|_{L^\infty(B_1)}).$$

Proof. By the conditions for F, there is a measureable coefficients $a_{ij}(x) \in \partial F(D^2u(x), rx)$ ($\partial F(M, x)$ is the subdifferential of F at (M, x)) such that

$$a_{ij}(x)\partial_{ij}\partial_e u(x) \le r\partial_e f(rx) - r(\partial_{x,e}F)(D^2u(x), rx) \quad \text{on } \Omega(u) \cap \{u < \psi\},$$

and

$$a_{ij}(x)\partial_{ij}u(x) \ge f(rx) \quad \text{on } \Omega(u) \cap \{u < \psi\},$$

where $\partial_{x,e}$ is the spatial directional derivative in the direction e.

Arguing by contradiction, suppose there is a point $y \in B_{1/2} \cap \Omega(u) \cap \{u < \psi\}$ such that $C\partial_e u(y) - u(y) < 0$. Define the auxiliary function

$$\phi(x) = C\partial_e u(x) - u(x) + \frac{c_0}{\lambda_1 n}|x - y|^2.$$

Then,

$$\begin{aligned} a_{ij}(x)\partial_{ij}\phi(x) &\le Cr\partial_e f(rx) - Cr(\partial_{x,e}F)(D^2u(x), rx) - f(rx) + 2c_0 \\ &\le Cr\|\nabla f\|_{L^\infty(B_1)} + Cr\|\nabla_x F\|_{L^\infty(B_M\times B_1)} - f(rx) + 2c_0 \le 0, \end{aligned}$$

on $B_{1/4}(y) \cap \Omega(u) \cap \{u < \psi\}$ for $r \le \tilde{r}_0 = \tilde{r}_0(C, c_0, \|\nabla f\|_{L^\infty(B_1)}, \|\nabla F\|_{L^\infty(B_M\times B_1)})$.

By the minimum principle of ϕ on $B_{1/4}(y) \cap \Omega(u) \cap \{u < \psi\}$, $\phi(y) < 0$ and $\phi \ge 0$ on $\partial(\Omega(u) \cap \{u < \psi\})$ ($C\partial_e\psi - \psi \ge 0$ in B_1), we have

$$\inf_{\partial B_{1/4}(y)\cap(\Omega(u)\cap\{u<\psi\})} \phi < 0 \quad \text{and} \quad \inf_{\partial B_{1/4}(y)\cap(\Omega(u)\cap\{u<\psi\})} (C\partial_e u - u) < -\frac{c_0}{32\lambda_1 n}.$$

Since $\epsilon_0 < \frac{c_0}{64\lambda_1 n}$, we have a contradiction. □

By the C^1-convergence of u_r, ψ_r to u_0, ψ_0, respectively, directional monotonicity for ψ and Lemma 5.3.3, we have the directional monotonicity for the solution $u \in P_1(M)$ with half-space type upper obstacle.

Lemma 5.3.4 (Directional monotonicity)**.** *Assume F satisfies (F1)-(F4) and F is C^1 and let $u \in P_1(M)$ and $f \geq c_0 > 0$ in B_1, $F(D^2\psi, x) \geq c_0 > 0$ in $\Omega(\psi)$. Let*

$$\psi_0 = \frac{a}{2}(x_1^+)^2$$

and

$$u_0(x) = \frac{1}{2}(x_1^+)^2 \quad \text{or} \quad u_0 = \frac{a}{2}(x_1^+)^2,$$

for some $1 \leq a$, where u_0 and ψ_0 are blowup functions of u and ψ, respectively. Then for any $\delta \in (0, 1]$ there exists

$$r_\delta = r_\delta(u, c_0, \|\nabla F\|_{L^\infty(B_M \times B_1)}, \|F\|_{L^\infty(B_M \times B_1)}, \|\nabla f\|_{L^\infty(B_1)}) > 0$$

such that

$$\begin{aligned} u &\geq 0 \quad \text{in } B_{r_1} \\ \partial_e u &\geq 0 \quad \text{in } B_{r_\delta} \quad \text{for any } e \in C_\delta \cap \partial B_1, \end{aligned}$$

where

$$C_\delta = \{x \in \mathbb{R}^n : x_1 > \delta |x'|\}, \quad x' = (x_2, ..., x_n).$$

Lemma 5.3.5. *Let u, ψ, F be as in Theorem 5.1.2. Then there exists $r_1 = r_1(u) > 0$ such that u is a solution of*

$$F(D^2u, x) = f\chi_{\{0<u<\psi\}} + F(D^2\psi, x)\chi_{\{0<u=\psi\}}, \quad 0 \leq u \leq \psi \quad \text{in } B_{r_1}.$$

Moreover, if u_0, ψ_0 are blowup functions of u, ψ at 0, then in an appropriate system of coordinates

$$\psi_0(x) = \frac{a}{2}(x_1^+)^2 \quad \text{and} \quad u_0(x) = \frac{1}{2}(x_1^+)^2.$$

Proof. By Lemma 5.3.4, there is $r_1 = r_1(u, c_0, \|\nabla F\|_{L^\infty(B_M \times B_1)}, \|F\|_{L^\infty(B_M \times B_1)}, \|\nabla f\|_{L^\infty(B_1)}) > 0$ such that $u \geq 0$ in B_{r_1}. Hence u is a solution of

$$F(D^2u, x) = f\chi_{\{0<u<\psi\}} + F(D^2\psi, x)\chi_{\{0<u=\psi\}}, \quad 0 \leq u \leq \psi \quad \text{in } B_{r_1}.$$

and $v := \psi - u$ is a solution of

$$F(D^2 v, x) = f\chi_{\{0<v<\psi\}} + F(D^2\psi, x)\chi_{\{0<v=\psi\}}, \quad 0 \le v \le \psi \quad \text{in } B_{r_1}.$$

By the conditions, $0 \le v \le \psi$, we have that $\{v > 0\} \subset \{\psi > 0\} = \Omega(\psi)$. Thus the condition, $F(D^2\psi, x), F(D^2\psi, x) - f \ge c_0 > 0$ in $\Omega(\psi)$ implies

$$F(D^2 v, x) = (F(D^2\psi, x) - f)\chi_{\{0<v<\psi\}} + F(D^2\psi, x)\chi_{\{0<v=\psi\}} \ge c_0 > 0 \quad \text{in } \{v > 0\}$$

and the non-degeneracy for v, i.e.,

$$\sup_{\partial B_r(x)} v \ge v(x) + \frac{\lambda}{8n} r^2 \quad x \in \overline{\Omega(v)} \cap B_{r_0},$$

where $B_r(x) \Subset B_{r_0}$. This implies $0 \in \Gamma(v_0) = \Gamma^{\psi_0}(u_0)$, where v_0 is a blowup functions of v at 0 such that $v_0 = \psi_0 - u_0$. Consequently, by the classification of global solution for the single obstacle problem for ψ and for the double obstacle problem (Theorem 5.3.2), we have

$$\psi_0(x) = \frac{a}{2}(x_1^+)^2, \quad u_0(x) = \frac{1}{2}(x_1^+)^2$$

in an appropriate system of coordinates. □

By Lemma 5.3.4 and Lemma 5.3.5, we have the uniqueness of blowup.

Proposition 5.3.6 (Uniqueness of blowup)**.** *Let u, ψ, F be as in Theorem 5.1.2. Then the blowup function of u at 0 is unique, i.e., in an appropriate system of coordinates, for any sequence $\lambda \to 0$,*

$$u_\lambda \to u_0 = \frac{1}{2}(x_1^+)^2 \quad \textit{in } C^{1,\alpha}_{loc}(\mathbb{R}^n)$$

as $\lambda \to 0$.

Lemma 5.3.7. *Let u, ψ, F and r_1 be as in Theorem 5.1.2. Then there is $r_1' = r_1'(u, \psi) > 0$ such that the blowup function of u at $x \in \Gamma(u) \cap B_{r_1'}$ are half-space functions.*

Proof. By Lemma 5.3.4 and 5.3.5, we have the directional monotonicity for u and ψ and the sign condition $u \ge 0$ in B_{r_1} for some $r_1 = r_1(u, c_0, \|\nabla F\|_{L^\infty(B_M \times B_1)}, \|F\|_{L^\infty(B_M \times B_1)}, \|\nabla f\|_{L^\infty(B_1)}) > 0$. Thus u is a solution of

$$F(D^2 u, x) = f\chi_{\{0<u<\psi\}} + F(D^2\psi, x)\chi_{\{0<u=\psi\}} \quad 0 \le u \le \psi \quad \text{in } B_{r_1}$$

and for any $\delta \in (0, 1]$, there exists

$$r_\delta = r_\delta(u, c_0, \|\nabla F\|_{L^\infty(B_M\times B_1)}, \|F\|_{L^\infty(B_M\times B_1)}, \|\nabla f\|_{L^\infty(B_1)}) > 0$$

such that $r_1 \geq r'_\delta = r'_\delta(u, \psi) > 0$ such that

$$\begin{aligned} \psi, u \geq 0 \quad & \text{in } B_{r'_1} \\ \partial_e \psi, \partial_e u \geq 0 \quad & \text{in } B_{r'_\delta} \quad \text{for any } e \in C_\delta. \end{aligned}$$

Then the free boundaries $\partial\{u = 0\} \cap B_{r'_1} = \Gamma(u) \cap B_{r'_1}, \partial\{\psi = 0\} \cap B_{r'_1}$ are represented by Lipschitz functions.

Case1) Let $x^0 \in \Gamma(u) \cap B_{r'_1}$ and assume that there exists $r_0 > 0$ such that

$$\{u = \psi\} \cap B_r(x^0) \neq \emptyset \quad \text{for all } r < r_0.$$

Then, there is a sequence of points, x^j such that $x^j \in \{u = \psi\}$ and $x^j \to x^0$ as $j \to \infty$. Then we have

$$\psi(x^j) = u(x^j) \to 0 \quad \text{as } j \to \infty,$$

i.e., $x^0 \in \{\psi = 0\}$. Furthermore, $\{\psi = 0\} \subset \{u = 0\}$ and $x^0 \in \Gamma(u)$ implies $x^0 \in \partial\{\psi = 0\}$. Then, the Lipschitz regularity of $\{\psi = 0\}$ and the positivity of u imply the uniform thickness assumption for $\{\psi = 0\}$ near x_0,

$$\delta_r(\psi, z) \geq \epsilon_0 \quad \text{for all } r < 1/4, x_0 \in \Gamma(u) \cap B_r(x_0), \text{ for some } \epsilon_0, \tilde{r} = \tilde{r}(x^0) > 0,$$

and

$$\delta_r(u, \psi, x_0) = \delta_r(\psi, x_0) \geq \epsilon_0 \quad \text{for all } r < 1/4.$$

Then, by the classification of the global solution, Proposition 5.3.2, we know that the blowup of u at x^0 is a half-space solution.

Case 2) Let $x^0 \in \Gamma(u)$ and assume that there exists $r_0 > 0$ such that

$$\{u = \psi\} \cap B_{r_0}(x^0) = \emptyset.$$

Then u is a solution of

$$F(D^2 u, x) = f\chi_{\{u>0\}}, \quad u \geq 0 \quad \text{in } B_{r_0}(x^0).$$

On the other hand, Lipschitz regularities of $\Gamma(u)$ implies the uniform thickness assumption for u near x_0. Then, the blowup function of u at 0 is a half-space solution. □

We have obtained important lemmas to prove the main theorem and the lest of proof follows arguments in [Lee98, PSU, IM16a, LPS]. Hence we present the sketch of the proof of Theorem 5.1.2.

Proof of Theorem 5.1.2. The directional monotonicity for u, Lemma 5.3.4 implies that the free boundary $\Gamma(u) \cap B_{r\delta/2}$ is a Lipschitz graph $x_n = f(x')$ with Lipschitz constant of f not exceeding δ. Since $\delta > 0$ is arbitrary, we have a tangent plane of $\Gamma(u)$ and the normal vector e_n at 0. By Lemma 5.3.7, we know that every point $z \in \Gamma(u) \cap B_{r_1'}$ has a tangent plane. Moreover again, by using the directional monotonicity, we obtain that $\Gamma(u) \cap B_{r_1'}$ is C^1. $\square$

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국문초록

이 박사학위 논문에서는 장애물 문제의 해의 정칙성과 자유경계의 정칙성을 다룬다. 특별히, 비 볼록 완전 비선형연산자의 장애물문제의 자유경계의 정칙성과 이중장애물 문제의 해의 정칙성과 자유경계의 정칙성을 다룬다.

비 볼록 완전 비선형연산자의 장애물문제의 자유경계의 정칙성을 증명하기 위해서, $F(D^2u) = 0$의 해의 내부 $C^{2,\alpha}$ 정칙성을 보였다. 라플라시안의 이중장애물 문제에서는 ACF 단조공식과 바이스 단조공식을 이용하였다. 이 단조공식은 완전 비선형연산자의 이중 장애물 문제에는 적용 될 수 없다. 그래서, 반공간 함수 $\psi = c(x_n^+)^2$를 위 장애물로 갖는 대역해 u에 대해 e가 e_n과 수직인 방향일 때, $\partial_e u/x_n$이 유한하다는 것을 이용하였다.

주요 어휘 : 장애물문제, 자유경계, 최적($C^{1,1}$) 정칙성, 비 복록 완전 비선형연산자, 이중장애물 문제

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